Technical Report Visibility Contractors

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Visibility Contractors

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Contents

Chapter 1

General Definitions

In the later, an obstacle ε_j is defined as a closed subset of $\mathbb{R}^n$.

1.1 Visibility relations between two points

Definition 1.1.1 *Let $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2 \in \mathbb{R}^n$ be two points, and ε_j be an obstacle . The visibility relation between the two points regards to the obstacle is defined as*

$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j = \emptyset, \tag{1.1}$$

with $Seg(\mathbf{x}_1, \mathbf{x}_2)$ the segment defined by the two edges $\mathbf{x}_1$ and $\mathbf{x}_2$.

The complement of this relation, the non-visibility relation, is denoted

$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)^c_{\varepsilon_j} = (\mathbf{x}_1 \overline{\mathrm{V}} \mathbf{x}_2)_{\varepsilon_j}. \tag{1.2}$$

Remark 1.1.1 *Some remarks about this relation:*

- *the visibility relation is reflexive*
$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow (\mathbf{x}_2 \mathrm{V} \mathbf{x}_1)_{\varepsilon_j}. \tag{1.3}$$
- *the visibility relation is symmetric*
$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_1)_{\varepsilon_j}. \tag{1.4}$$
- *the visibility relation is not transitive (Figure 1.1)*
$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j} \ \wedge \ (\mathbf{x}_2 \mathrm{V} \mathbf{x}_3)_{\varepsilon_j} \not\Rightarrow (\mathbf{x}_1 \mathrm{V} \mathbf{x}_3)_{\varepsilon_j}. \tag{1.5}$$
- *the non-visibility relation can be noted*
$$(\mathbf{x}_1 \overline{\mathrm{V}} \mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j \neq \emptyset. \tag{1.6}$$

Figure 1.1 presents visibility and non-visibility examples between two points regards to an obstacle.

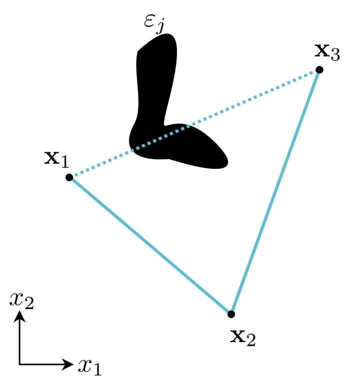
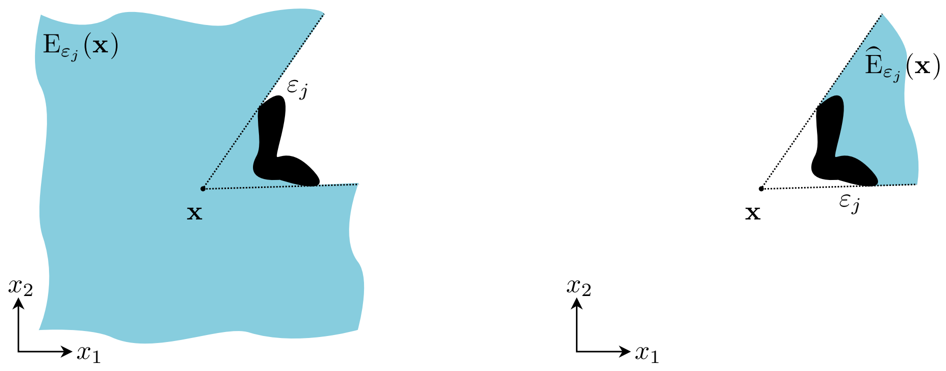


Figure 1.1: In this example: $(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j}$, $(\mathbf{x}_2 \mathrm{V} \mathbf{x}_3)_{\varepsilon_j}$ and $(\mathbf{x}_1 \overline{\mathrm{V}} \mathbf{x}_3)_{\varepsilon_j}$.



(a) Visible space example.

(b) Non-visible space example.

Figure 1.2: Visibility and non-visibility spaces example.

1.2 Visible and non-visible spaces of a point

Definition 1.2.1 *Let $\mathbf{x} \in \mathbb{R}^n$ be a point and ε_j an obstacle, with $\mathbf{x} \notin \varepsilon_j$.*
The visible space of the point $\mathbf{x}$ regards to the obstacle ε_j is defined as

$$\mathrm{E}_{\varepsilon_j}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\varepsilon_j}\}. \tag{1.7}$$

The non-visible space of the point $\mathbf{x}$ regards to the obstacle ε_j is defined as

$$\widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid (\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_i)_{\varepsilon_j}\}. \tag{1.8}$$

Remark 1.2.1

$$\mathrm{E}^c_{\varepsilon_j}(\mathbf{x}) = \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}). \tag{1.9}$$

Remark 1.2.2

$$(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow \mathbf{x}_1 \in \mathrm{E}_{\varepsilon_j}(\mathbf{x}_2), \tag{1.10}$$
$$(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow \mathbf{x}_2 \in \mathrm{E}_{\varepsilon_j}(\mathbf{x}_1), \tag{1.11}$$
$$\mathbf{x}_2 \in \mathrm{E}_{\varepsilon_j}(\mathbf{x}_1) \Leftrightarrow \mathbf{x}_1 \in \mathrm{E}_{\varepsilon_j}(\mathbf{x}_2). \tag{1.12}$$

Remark 1.2.3

$$(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow \mathbf{x}_1 \in \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}_2), \tag{1.13}$$
$$(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j} \Leftrightarrow \mathbf{x}_2 \in \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}_1), \tag{1.14}$$
$$\mathbf{x}_2 \in \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}_1) \Leftrightarrow \mathbf{x}_1 \in \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}_2). \tag{1.15}$$

Figures 1.2a and 1.2b present examples of visible and non-visible spaces.

1.3 Visible/non-visible/partially-visible spaces of a closed set

Definition 1.3.1 *Let $\mathbb{X} \subset \mathbb{R}^n$ be a closed set and ε_j an obstacle, with $\mathbb{X} \cap \varepsilon_j = \emptyset$.*
The visible space of $\mathbb{X}$ regards to ε_j is defined as

$$\mathrm{E}_{\varepsilon_j}(\mathbb{X}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_j}\}. \tag{1.16}$$

The non-visible space of $\mathbb{X}$ regards to ε_j is defined as

$$\widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}\}. \tag{1.17}$$

The partially-visible space of $\mathbb{X}$ regards to ε_j is defined as

$$\widetilde{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid \exists \mathbf{x}_1 \in \mathbb{X}, \exists \mathbf{x}_2 \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x}_1)_{\varepsilon_j} \;\wedge\; (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j}\}. \tag{1.18}$$

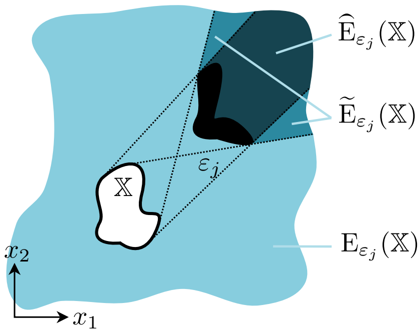


Figure 1.3: Example of visible, non-visible and partially-visible spaces.

It can be noticed that the visibility relation is a binary relation when considering points, but a ternary relation when considering sets. The partially-visible space corresponds classically to the twilight (compared to the light (visible space) and the shade (non-visible space)). Figure 1.3 presents an example of visible, non-visible and partially-visible spaces.

Proposition 1.3.1 links the visibility of a set to the visibility of its points.

Proposition 1.3.1 *Let $\mathbb{X} \subset \mathbb{R}^n$ be a closed set, ε_j an obstacle with $\mathbb{X} \cap \varepsilon_j = \emptyset$, $\mathbf{x} \in \mathbb{X}$ and $\mathbf{x}_i \in \mathbb{R}^n$ two points with $\mathbf{x}_i \notin \mathbb{X}$. Then*

$$(\mathbf{x}_i \mathrm{V} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \in \mathrm{E}_{\varepsilon_j}(\mathbb{X}) \cup \widetilde{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}), \tag{1.19}$$

$$(\mathbf{x}_i \overline{\mathrm{V}} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) \cup \widetilde{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}). \tag{1.20}$$

Proof

$$\begin{aligned}
&(\mathbf{x}_i \mathrm{V} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \notin \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) \text{ (Eq. 1.16)} \\
&(\mathbf{x}_i \mathrm{V} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \in \widetilde{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) \cup \mathrm{E}_{\varepsilon_j}(\mathbb{X}) \\
&(\mathbf{x}_i \overline{\mathrm{V}} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \notin \mathrm{E}_{\varepsilon_j}(\mathbb{X}) \text{ (Eq. 1.16)} \\
&(\mathbf{x}_i \overline{\mathrm{V}} \mathbf{x})_{\varepsilon_j} \Rightarrow \mathbf{x}_i \in \widetilde{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) \cup \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X})
\end{aligned}$$

1.4 Visibility regards to a set of obstacles

An *environment* of $\mathbb{R}^n$, noted $\mathcal{E}$, is defined as a set of n_O obstacles

$$\mathcal{E} = \bigcup_{j=1}^{n_O} \varepsilon_j, \tag{1.21}$$

with $\varepsilon_1, \cdots, \varepsilon_j, \cdots, \varepsilon_{n_O}$ obstacles of $\mathbb{R}^n$.

It is possible to extend the previous definitions to an environment:

$$(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\mathcal{E}} \Leftrightarrow Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \mathcal{E} = \emptyset, \tag{1.22}$$

$$(\mathbf{x}_1 \overline{\mathrm{V}} \mathbf{x}_2)_{\mathcal{E}} \Leftrightarrow Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \mathcal{E} \neq \emptyset, \tag{1.23}$$

$$\mathrm{E}_{\mathcal{E}}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid (\mathbf{x}_i \mathrm{V} \mathbf{x})_{\mathcal{E}}\}, \tag{1.24}$$

$$\mathrm{E}^c_{\mathcal{E}}(\mathbf{x}) = \widehat{\mathrm{E}}_{\mathcal{E}}(\mathbf{x}), \tag{1.25}$$

$$\mathrm{E}_{\mathcal{E}}(\mathbb{X}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x} \mathrm{V} \mathbf{x}_i)_{\mathcal{E}}\}, \tag{1.26}$$

$$\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{X}) = \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x} \overline{\mathrm{V}} \mathbf{x}_i)_{\mathcal{E}}\}. \tag{1.27}$$

The objective then is to characterize the visibility regards to an environment by considering the visibility regards to its obstacles.

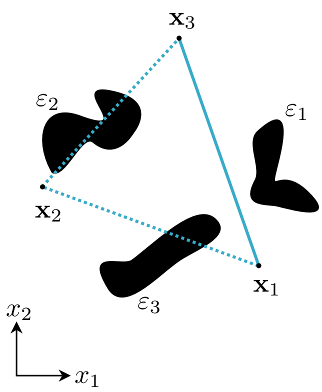


Figure 1.4: In this example $(\mathbf{x}_1\mathrm{V}\mathbf{x}_3)_{\varepsilon_1}$ and $(\mathbf{x}_1\mathrm{V}\mathbf{x}_3)_{\varepsilon_2}$ and $(\mathbf{x}_1\mathrm{V}\mathbf{x}_3)_{\varepsilon_3}$, then it can be conclude that $(\mathbf{x}_1\mathrm{V}\mathbf{x}_3)_{\mathcal{E}}$. It can also be noticed that $(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_1}$ and $(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_2}$ and $(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_3}$, which leads to $(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\mathcal{E}}$. The same idea applies to $\mathbf{x}_2$ and $\mathbf{x}_3$: $(\mathbf{x}_2\mathrm{V}\mathbf{x}_3)_{\varepsilon_1}$ and $(\mathbf{x}_2\overline{\mathrm{V}}\mathbf{x}_3)_{\varepsilon_2}$ and $(\mathbf{x}_2\mathrm{V}\mathbf{x}_3)_{\varepsilon_3}$, then $(\mathbf{x}_2\overline{\mathrm{V}}\mathbf{x}_3)_{\mathcal{E}}$.

Proposition 1.4.1 *Let $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2 \in \mathbb{R}^n$ be two distinct points, and $\mathcal{E}$ an environment of $\mathbb{R}^n$ composed by n_O obstacles with $\mathbf{x}_1 \notin \mathcal{E}$ and $\mathbf{x}_2 \notin \mathcal{E}$. Then*

$$(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\mathcal{E}} \Leftrightarrow \bigwedge_{j=1}^{n_O} (\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_j}, \tag{1.28}$$

$$(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\mathcal{E}} \Leftrightarrow \bigvee_{j=1}^{n_O} (\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j}. \tag{1.29}$$

Proof (Eq. 1.28)

$$\begin{aligned}
(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\mathcal{E}} &\Leftrightarrow Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \mathcal{E} = \emptyset \text{ (Eq. 1.22)},\\
&\Leftrightarrow Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \Big(\bigcup_{j=1}^{n_O} \varepsilon_j\Big) = \emptyset \text{ (Eq. 1.21)},\\
&\Leftrightarrow \bigcup_{j=1}^{n_O} (Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \varepsilon_j) = \emptyset,\\
&\Leftrightarrow \forall \varepsilon_j \in \mathcal{E}, Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \varepsilon_j = \emptyset,\\
&\Leftrightarrow \forall \varepsilon_j \in \mathcal{E}, (\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_j} \text{ (Eq. 1.1)},\\
(\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\mathcal{E}} &\Leftrightarrow \bigwedge_{j=1}^{n_O} (\mathbf{x}_1\mathrm{V}\mathbf{x}_2)_{\varepsilon_j}.
\end{aligned}$$

Proof (Eq. 1.29)

$$\begin{aligned}
(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\mathcal{E}} &\Leftrightarrow Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \mathcal{E} \neq \emptyset \text{ (Eq. 1.23)},\\
&\Leftrightarrow Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \Big(\bigcup_{j=1}^{n_O} \varepsilon_j\Big) \neq \emptyset \text{ (Eq. 1.21)},\\
&\Leftrightarrow \bigcup_{j=1}^{n_O} (Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \varepsilon_j) \neq \emptyset,\\
&\Leftrightarrow \exists \varepsilon_j \in \mathcal{E} | Seg(\mathbf{x}_1,\mathbf{x}_2) \cap \varepsilon_j \neq \emptyset,\\
&\Leftrightarrow \exists \varepsilon_j \in \mathcal{E} | (\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j} \text{ (Eq. 1.6)},\\
(\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\mathcal{E}} &\Leftrightarrow \bigvee_{j=1}^{n_O} (\mathbf{x}_1\overline{\mathrm{V}}\mathbf{x}_2)_{\varepsilon_j}.
\end{aligned}$$

Figure 1.4 illustrates Propositions 1.4.1.

It is also possible to characterize the visibility spaces of a point regards to an environment to the visibility spaces of this point regards to the obstacles of the environment.

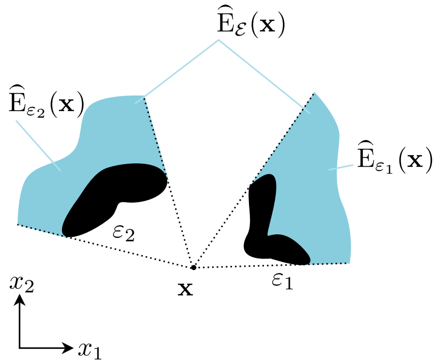


Figure 1.5: In this example $\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbf{x}) = \widehat{\mathrm{E}}_{\varepsilon_1}(\mathbf{x}) \cup \widehat{\mathrm{E}}_{\varepsilon_2}(\mathbf{x})$.

Proposition 1.4.2 *Let $\mathbf{x} \in \mathbb{R}^n$ be a point and $\mathcal{E}$ an environment composed by n_O obstacles with $\mathbf{x} \notin \mathcal{E}$. Then*

$$\mathrm{E}_{\mathcal{E}}(\mathbf{x}) = \bigcap_{j=1}^{n_O} \mathrm{E}_{\varepsilon_j}(\mathbf{x}), \tag{1.30}$$

$$\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbf{x}) = \bigcup_{j=1}^{n_O} \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}). \tag{1.31}$$

Proof (Eq. 1.30)

$$\begin{aligned}
\mathrm{E}_{\mathcal{E}}(\mathbf{x}) &= \{\mathbf{x}_i \in \mathbb{R}^n | (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\mathcal{E}}\} \text{ (Eq. 1.24)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^n | \bigwedge_{j=1}^{n_O} (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\varepsilon_j}\} \text{ (Eq. 1.28)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^n | (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\varepsilon_1}\} \cap \dots \cap \{\mathbf{x}_i \in \mathbb{R}^n | (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\varepsilon_j}\} \cap \dots \cap \{\mathbf{x}_i \in \mathbb{R}^n | (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\varepsilon_{n_O}}\},\\
&= \mathrm{E}_{\varepsilon_1}(\mathbf{x}) \cap \dots \cap \mathrm{E}_{\varepsilon_j}(\mathbf{x}) \cap \dots \cap \mathrm{E}_{\varepsilon_{n_O}}(\mathbf{x}) \text{ (Eq. 1.7)},\\
\mathrm{E}_{\mathcal{E}}(\mathbf{x}) &= \bigcap_{j=1}^{n_O} \mathrm{E}_{\varepsilon_j}(\mathbf{x}).
\end{aligned}$$

Proof (Eq. 1.16)

$$\begin{aligned}
\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbf{x}) &= \mathrm{E}^c_{\mathcal{E}}(\mathbf{x}) \text{ (Eq. 1.25)},\\
&= \Big(\bigcap_{j=1}^{n_O} \mathrm{E}_{\varepsilon_j}(\mathbf{x})\Big)^c \text{ (Eq. 1.30)},\\
&= \bigcup_{j=1}^{n_O} \mathrm{E}^c_{\varepsilon_j}(\mathbf{x}),\\
\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbf{x}) &= \bigcup_{j=1}^{n_O} \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbf{x}) \text{ (Eq. 1.9)}.
\end{aligned}$$

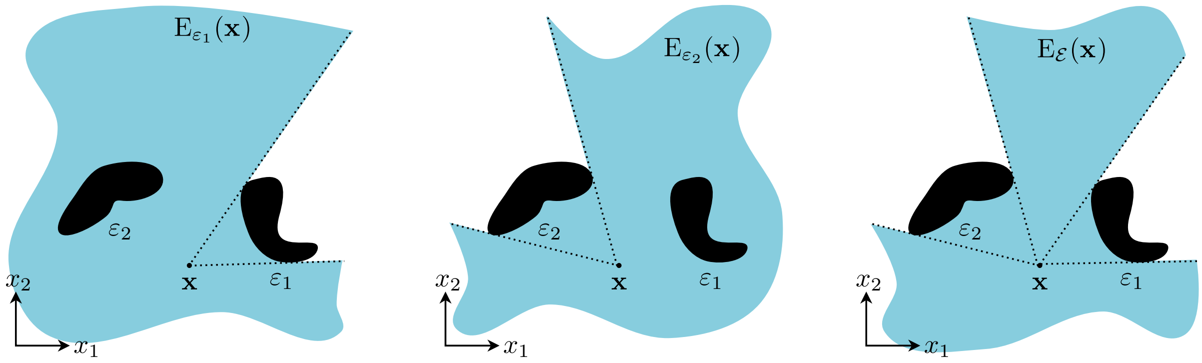
Figures 1.5 and 1.6 illustrate Proposition 1.4.2.

Those propositions can be generalized to closed sets. It is possible to characterize the visibility spaces of a closed set regards to an environment by the visibility spaces of this set regards to the obstacles of the environment.

Proposition 1.4.3 *Let $\mathbb{X} \subset \mathbb{R}^n$ be a closed set and $\mathcal{E}$ an environment defined by n_O obstacles with $\mathbb{X} \cap \mathcal{E} = \emptyset$. Then*

$$\mathrm{E}_{\mathcal{E}}(\mathbb{X}) = \bigcap_{j=1}^{n_O} \mathrm{E}_{\varepsilon_j}(\mathbb{X}), \tag{1.32}$$

$$\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{X}) \supseteq \bigcup_{j=1}^{n_O} \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}). \tag{1.33}$$



(a) $\mathrm{E}_{\varepsilon_1}(\mathbf{x})$. (b) $\mathrm{E}_{\varepsilon_2}(\mathbf{x})$ (c) $\mathrm{E}_{\mathcal{E}}(\mathbf{x})$

Figure 1.6: In this example $\mathrm{E}_{\mathcal{E}}(\mathbf{x}) = \mathrm{E}_{\varepsilon_1}(\mathbf{x}) \cap \mathrm{E}_{\varepsilon_2}(\mathbf{x})$.

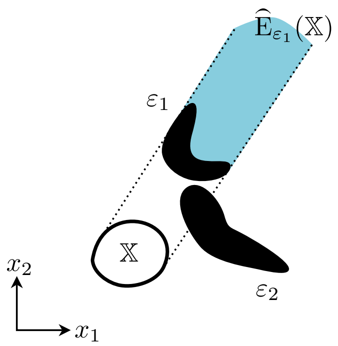
Proof (Eq. 1.32)

$$\begin{aligned}
\mathrm{E}_{\mathcal{E}}(\mathbb{X}) =& \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}\mathrm{V}\mathbf{x}_i)_{\mathcal{E}}\} \text{ (Eq. 1.26)},\\
=& \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, \bigwedge_{j=1}^{n_O} (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_j}\} \text{ (Eq. 1.28)},\\
=& \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_1}\} \cap \dots \cap \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_j}\} \cap \cdots\\
& \cdots \cap \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_{n_O}}\},\\
=& \bigcap_{j=1}^{n_O} \{x_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\mathrm{V}\mathbf{x})_{\varepsilon_j}\},\\
\mathrm{E}_{\mathcal{E}}(\mathbb{X}) =& \bigcap_{j=1}^{m} \mathrm{E}_{\mathcal{E}_j}(\mathbb{X}) \text{ (Eq. 1.16)}.
\end{aligned}$$

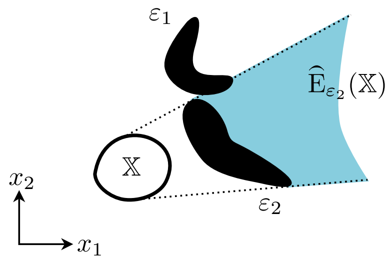
Proof (Eq. 1.33)

$$\begin{aligned}
\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{X}) &= \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\mathcal{E}}\} \text{ (Eq. 1.27)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, \bigvee_{j=1}^{n_O} (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}\} \text{ (Eq. 1.1)}.\\
\bigcup_{j=1}^{n_O} \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}) &= \bigcup_{j=1}^{n_O} \{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}\} \text{ (Eq. 1.17)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^n \mid \bigvee_{j=1}^{n_O} [\forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}]\}.\\
\{\mathbf{x}_i \in \mathbb{R}^n \mid \forall \mathbf{x} \in \mathbb{X}, \bigvee_{j=1}^{n_O} (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}\} &\supseteq \{\mathbf{x}_i \in \mathbb{R}^n \mid \bigvee_{j=1}^{n_O} [\forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i\overline{\mathrm{V}}\mathbf{x})_{\varepsilon_j}]\},\\
\Rightarrow \widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{X}) &\supseteq \bigcup_{j=1}^{n_O} \widehat{\mathrm{E}}_{\varepsilon_j}(\mathbb{X}).
\end{aligned}$$

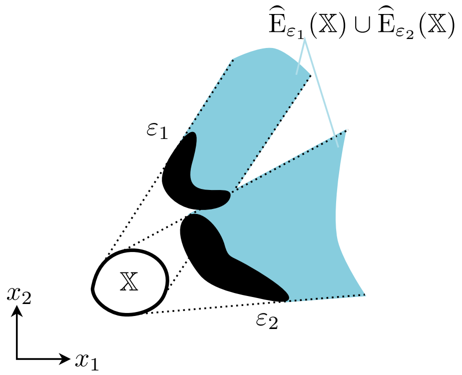
It can be noticed that the non-visible space of a closed set regards to an environment can not be perfectly characterized by the non-visible spaces of this set regards to the obstacles of the environment (Equation 1.33). It can only be under approximated, this is illustrated Figure 1.7.



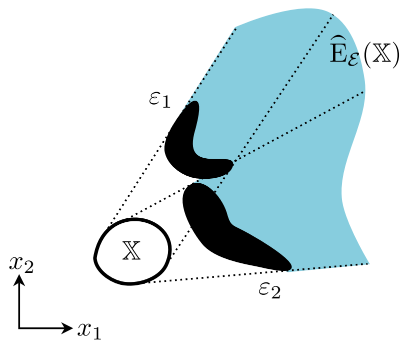
(a) $\widehat{\mathrm{E}}_{\varepsilon_1}(\mathbb{X})$.



(b) $\widehat{\mathrm{E}}_{\varepsilon_2}(\mathbb{X})$



(c) $\widehat{\mathrm{E}}_{\varepsilon_1}(\mathbb{X}) \cup \widehat{\mathrm{E}}_{\varepsilon_2}(\mathbb{X})$.



(d) $\mathrm{E}_{\mathcal{E}}(\mathbb{X})$

Figure 1.7: In this example it can be noticed that $\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{X}) \supseteq \widehat{\mathrm{E}}_{\varepsilon_1}(\mathbb{X}) \cup \widehat{\mathrm{E}}_{\varepsilon_2}(\mathbb{X})$.

Chapter 2

Particular cases of the visibility

In this section the visibility of three types of sources of $\mathbb{R}^2$ are considered (point, segment and box - defined later -) regards to two types of obstacles of $\mathbb{R}^2$ (segment and convex polygon). The interest is that in those cases the visibility spaces can be defined by sets of inequalities.

For a segment as obstacle, the following notation is used

$$\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j}), \tag{2.1}$$

with $\mathbf{e}_{1_j} \in \mathbb{R}^2$ and $\mathbf{e}_{2_j} \in \mathbb{R}^2$ two distinct points that represent the edges of the segment ε_j^s.

ε_j^p is an obstacle that corresponds to a convex polygon defined by n_{P_j} edges, named $\mathbf{e}_1, \cdots, \mathbf{e}_k, \cdots, \mathbf{e}_{n_{P_j}}$, in a trigonometric order, with $\mathbf{e}_k = (e_{1_k}, e_{2_k}) \in \mathbb{R}^2$.

ε_j^p can also be associated to a closed environment composed by n_{P_j} segments:

$$\varepsilon_j^p \equiv \bigcup_{k=1}^{n_{P_j}} Seg(\mathbf{e}_k, \mathbf{e}_{k+1}), \text{ with } \mathbf{e}_{n_{P_j}+1} \equiv \mathbf{e}_1, \tag{2.2}$$

In the following it is noted

$$Seg(\mathbf{e}_k, \mathbf{e}_{k+1}) = \varepsilon_k^s. \tag{2.3}$$

The interest of considering convex polygons remains in Proposition 2.0.4. Note that with random obstacles it is an inclusion, not an equality (Equation 1.33).

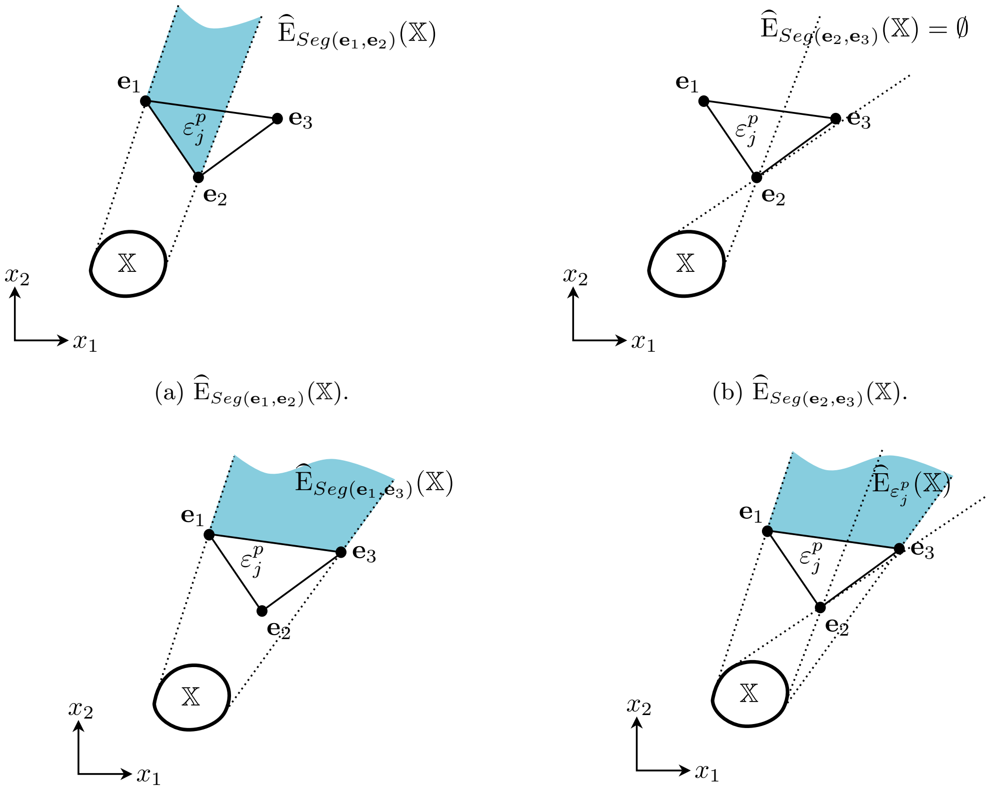
Proposition 2.0.4 *Let $\mathbb{X} \in \mathbb{R}^2$ be a closed set and ε_j^p an obstacle composed by n_{P_j} edges with $\mathbb{X} \cap \varepsilon_j^p = \emptyset$. Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbb{X}) = \bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_k^s}(\mathbb{X}). \tag{2.4}$$

Proof

$$\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbb{X}) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i \overline{\nabla} \mathbf{x})_{\varepsilon_j^p}\} \text{ (Eq. 1.17)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, Seg(\mathbf{x}_i, \mathbf{x}) \cap \varepsilon_j^p \neq \emptyset\} \text{ (Eq. 1.6)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, Seg(\mathbf{x}_i, \mathbf{x}) \cap \Big(\bigcup_{k=1}^{n_{P_j}} \varepsilon_k^s\Big) \neq \emptyset\} \text{ (Eq. 2.2)},\\
&= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, \bigcup_{k=1}^{n_{P_j}} (Seg(\mathbf{x}_i, \mathbf{x}) \cap \varepsilon_k^s) \neq \emptyset\},\\
&= \bigcup_{k=1}^{n_{P_j}} \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, Seg(\mathbf{x}_i, \mathbf{x}) \cap \varepsilon_k^s \neq \emptyset\},\\
&= \bigcup_{k=1}^{n_{P_j}} \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in \mathbb{X}, (\mathbf{x}_i \overline{\nabla} \mathbf{x}))_{\varepsilon_k^s}\} \text{ (Eq. 1.6)},\\
\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbb{X}) &= \bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_k^s}(\mathbb{X}) \text{ (Eq. 1.17)}.
\end{aligned}$$

Figure 2.1 illustrates Proposition 2.0.4.



(a) $\widehat{\mathrm{E}}_{Seg(\mathbf{e}_1,\mathbf{e}_2)}(\mathbb{X})$.

(b) $\widehat{\mathrm{E}}_{Seg(\mathbf{e}_2,\mathbf{e}_3)}(\mathbb{X})$.

(c) $\widehat{\mathrm{E}}_{Seg(\mathbf{e}_1,\mathbf{e}_3)}(\mathbb{X})$.

(d) $\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbb{X})$.

Figure 2.1: Non-visible space of a closed set $\mathbb{X}$ regards to a convex polygon ε_j^p

2.1 Visibility spaces of a point

2.1.1 Regards to a segment obstacle

Proposition 2.1.1 *Let* $\mathbf{x} \in \mathbb{R}^2$ *be a point and* $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ *an obstacle with* $\mathbf{x} \notin \varepsilon_j^s$. *Then*

$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid [\mathbf{x}_i \cup \mathbf{x}] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \;\vee\; \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \;\vee \\ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \;\vee\; \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0\},\end{aligned} \tag{2.5}$$

with

$$\zeta_x = \begin{cases} 1 & \textit{if } \det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \textit{otherwise.} \end{cases} \tag{2.6}$$

Proof

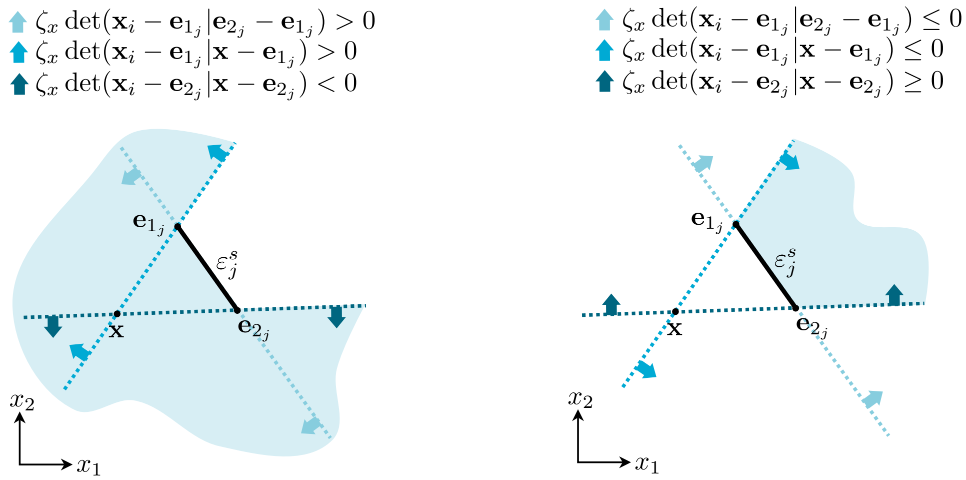
$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(\mathbf{x}) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid (\mathbf{x} \mathrm{V} \mathbf{x}_i)_{\varepsilon_j^s}\} \text{ (Eq. 1.7)}, \\ &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid Seg(\mathbf{x}, \mathbf{x}_i) \cap Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j}) = \emptyset\} \text{ (Eq. 1.1)}, \\ &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \cdot \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \quad \vee \\ &\qquad \det(\mathbf{e}_{1_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) \cdot \det(\mathbf{e}_{2_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) > 0 \quad \vee \\ &\qquad [\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\} \text{ (Eq. A.5)}.\end{aligned}$$

According to Proposition C.1.1 (Appendix):

$$\begin{aligned}&\det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \cdot \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \;\vee \\ &\quad \det(\mathbf{e}_{1_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) \cdot \det(\mathbf{e}_{2_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) > 0 \\ \Leftrightarrow &\zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \;\vee\; \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \;\vee \\ &\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0.\end{aligned} \tag{2.7}$$

Then

$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid [\mathbf{x}_i \cup \mathbf{x}] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \;\vee\; \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \;\vee \\ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \;\vee\; \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0\},\end{aligned}$$



(a) $\mathrm{E}_{\varepsilon_j^s}(\mathbf{x})$. (b) $\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x})$

Figure 2.2: Visible and non-visible spaces characterization example.

The following proposition characterizes the non-visible space of a point regards to a segment.

Proposition 2.1.2 *Let* $\mathbf{x} \in \mathbb{R}^2$ *be a point and* $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ *an obstacle with* $\mathbf{x} \notin \varepsilon_j^s$. *Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \begin{array}{lll} \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) & \leq 0 & \wedge \\ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) & \leq 0 & \wedge \\ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) & \geq 0 & \wedge \\ [\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] & \neq \emptyset & \}, \end{array} \tag{2.8}$$

with

$$\zeta_x = \begin{cases} 1 & \textit{if } \det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \textit{otherwise}. \end{cases}$$

Proof

$$\begin{aligned} \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}) &= (\mathrm{E}_{\varepsilon_j^s}(\mathbf{x}))^c \text{ (Eq. 1.9)}, \\ &= \Big(\{\mathbf{x}_i \in \mathbb{R}^2 \mid [\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0\}\Big)^c \text{ (Eq. 2.5)}, \\ &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \Big([\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0\Big)^c\}, \\ \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid [\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset \ \wedge \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \\ &\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0\}. \end{aligned}$$

Figure 2.2 illustrates Propositions 2.1.1 and 2.1.2.

2.1.2 Regards to a polygon obstacle

The following proposition characterizes the visible space of a point regards to a convex polygon.

Proposition 2.1.3 *Let* $\mathbf{x} \in \mathbb{R}^2$ *be a point and* ε_j^p *a convex polygon defined by* n_{P_j} *edges with* $\mathbf{x} \notin \varepsilon_j^p$. *Then*

$$\mathrm{E}_{\varepsilon_j^p}(\mathbf{x}) = \bigcap_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(\mathbf{x}). \tag{2.9}$$

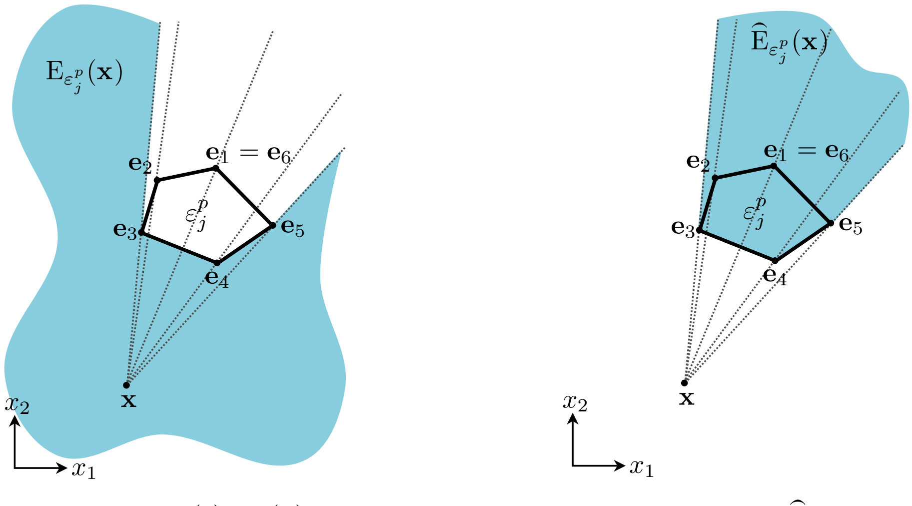


Figure 2.3: Visible and non-visible spaces of a point regards to a convex polygon.

Proof

$$\mathrm{E}_{\varepsilon_j^p}(\mathbf{x}) = \bigcap_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(\mathbf{x}) \text{ (Eq. 1.30 et 2.2).}$$

The following proposition characterizes the non-visible space of a point regards to a polygon.

Proposition 2.1.4 *Let $\mathbf{x} \in \mathbb{R}^2$ be a point and ε_j^p an obstacle defined by n_{P_j} edges with $\mathbf{x} \notin \varepsilon_j^p$. Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbf{x}) = \bigcup_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(\mathbf{x}). \tag{2.10}$$

Proof

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}(\mathbf{x}) = \bigcup_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(\mathbf{x}) \text{ (Eq. 1.31 et 2.2).}$$

Figure 2.3 illustrates Propositions 2.1.3 and 2.1.4.

2.2 Visibility of a segment

A segment can be considered as a closed subset of $\mathbb{R}^2$.

2.2.1 Regards to a segment obstacle

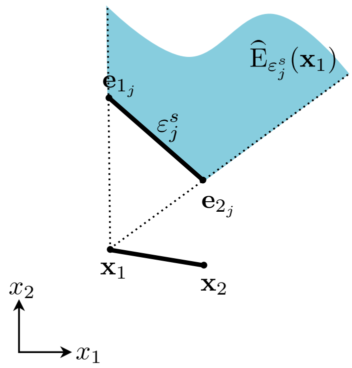
First we consider the non-visible space of a segment regards to an other segment. It can be noticed that this space can be characterized by the non-visible spaces of the segment edges.

Proposition 2.2.1 *Let $Seg(\mathbf{x}_1, \mathbf{x}_2)$ be a segment with $\mathbf{x}_1 \in \mathbb{R}^2$ and $\mathbf{x}_2 \in \mathbb{R}^2$, and $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ be an obstacle with $Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j^s = \emptyset$. Then*

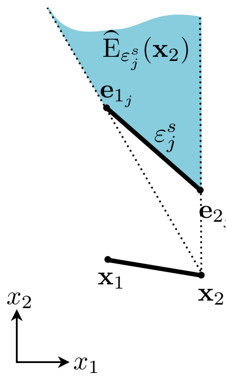
$$\widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2). \tag{2.11}$$

Proof

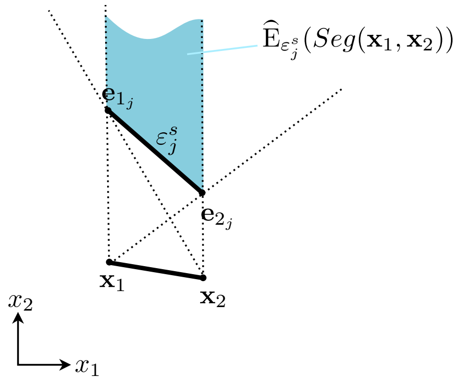
$$\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{x}_i \mathrm{V} \mathbf{x})_{\varepsilon_j^s}\} \text{ (Eq. 1.16)}, \\
&= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), [\mathbf{x} \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset \wedge \\
&\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0\} \text{(Eq. 2.8)},
\end{aligned}$$



(a) $\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1)$.



(b) $\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2)$.



(c) $\widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2)$.

Figure 2.4: Non-visible space of a segment regards to a segment.

$$
\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2) = &\{\mathbf{x}_i \in \mathbb{R}^2 \mid \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \wedge \\
&\quad [\mathbf{x}_1 \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset\} \cap \\
&\{\mathbf{x}_i \in \mathbb{R}^2 \mid \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \wedge \\
&\quad [\mathbf{x}_2 \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset\}, \text{ (Eq. 2.8)}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2) = &\{\mathbf{x}_i \in \mathbb{R}^2 \mid \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \wedge \\
&\quad [\mathbf{x}_1 \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset \wedge \\
&\quad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\quad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \wedge \\
&\quad [\mathbf{x}_2 \cup \mathbf{x}_i] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] \neq \emptyset\}.
\end{aligned}
$$

According to Proposition C.2.1 (Appendix)

$$
\begin{aligned}
&\forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\qquad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0 \\
&\Leftrightarrow \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \wedge \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \geq 0 \wedge \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \geq 0 \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \wedge \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0.
\end{aligned}
$$

Then

$$
\widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_2).
$$

Figure 2.4 illustrates Proposition 2.2.1.

The visible space of a segment regards to an other segment is more difficult to characterize. Indeed, it is not possible to characterize it by considering the visible spaces of the edges of the segment.

Proposition 2.2.2 *Let $Seg(\mathbf{x}_1, \mathbf{x}_2)$ be a segment with $\mathbf{x}_1 \in \mathbb{R}^2$ and $\mathbf{x}_2 \in \mathbb{R}^2$, and $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ be an* *obstacle with* $Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j^s = \emptyset$. *Then*

$$\begin{aligned}
&\mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \\
&\quad (\zeta_{x_1} = \zeta_{x_2}) \ \wedge \ \Big(\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \ \vee \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) > 0 \ \vee \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) < 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) < 0\Big) \ \vee \\
&\quad (\zeta_{x_1} = -\zeta_{x_2}) \ \wedge \ \Big(\\
&\qquad \big(\zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) < 0\big) \ \wedge \\
&\qquad \big(\zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \ \vee \ \zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) < 0\big)\Big) \ \vee \\
&\quad \Big([\mathbf{x}_i \cup \mathbf{x}_1 \cup \mathbf{x}_2] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\Big)\}.
\end{aligned} \tag{2.12}$$

with

$$\zeta_{x_1} = \begin{cases} 1 & \textit{if } \det(\mathbf{x}_1 - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \textit{otherwise.} \end{cases}$$

$$\zeta_{x_2} = \begin{cases} 1 & \textit{if } \det(\mathbf{x}_2 - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \textit{otherwise.} \end{cases}$$

$$\zeta_{e_1} = \begin{cases} 1 & \textit{if } \det(\mathbf{e}_{1_j} - \mathbf{x}_1 | \mathbf{x}_2 - \mathbf{x}_1) > 0, \\ -1 & \textit{otherwise.} \end{cases}$$

$$\zeta_{e_2} = \begin{cases} 1 & \textit{if } \det(\mathbf{e}_{2_j} - \mathbf{x}_1 | \mathbf{x}_2 - \mathbf{x}_1) > 0, \\ -1 & \textit{otherwise.} \end{cases}$$

Proof The proof of this proposition is presented in the appendix, Section C.3.

Figures 2.5 and 2.6 illustrate Proposition 2.2.2.

2.2.2 Regards to a polygon obstacle

As it has already been pointed out, a polygon can be considered as a set of segments. It is then possible to characterize the visibility spaces of a segment regards to a polygon by considering the visibility spaces of the segment regards to the segments that define the polygon.

Then, for the visible space of a segment regards to a polygon:

Proposition 2.2.3 *Let* $Seg(\mathbf{x}_1, \mathbf{x}_2)$ *be a segment with* $\mathbf{x}_1 \in \mathbb{R}^2$ *and* $\mathbf{x}_2 \in \mathbb{R}^2$, *and* ε_j^p *an convex polygon defined by* n_{P_j} *edges with* $Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j^p = \emptyset$. *Then*

$$\mathrm{E}_{\varepsilon_j^p}(Seg(\mathbf{x}_1,\mathbf{x}_2)) = \bigcap_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)). \tag{2.13}$$

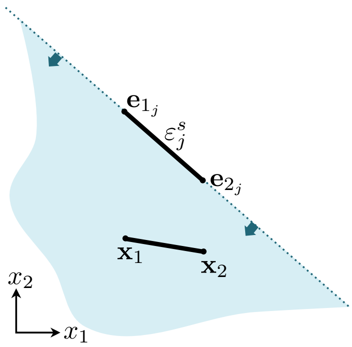
Proof

$$\mathrm{E}_{\varepsilon_j^p}(Seg(\mathbf{x}_1,\mathbf{x}_2)) = \bigcap_{k=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_k^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)) \text{ (Eq. 1.32 et 2.2).}$$

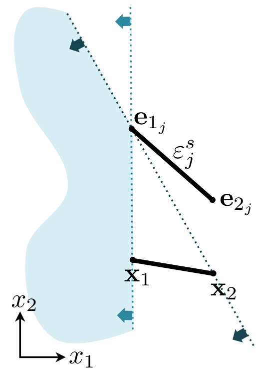
And for the non-visible space of a segment regards to a polygon:

Proposition 2.2.4 *Let* $Seg(\mathbf{x}_1, \mathbf{x}_2)$ *be a segment with* $\mathbf{x}_1 \in \mathbb{R}^2$ *and* $\mathbf{x}_2 \in \mathbb{R}^2$, *and* ε_j^p *a convex polygon defined by* n_{P_j} *edges with* $Seg(\mathbf{x}_1, \mathbf{x}_2) \cap \varepsilon_j^p = \emptyset$. *Then*

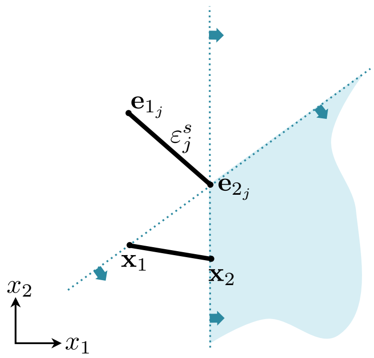
$$\widehat{\mathrm{E}}_{\varepsilon_j^p}(Seg(\mathbf{x}_1,\mathbf{x}_2)) = \Big(\bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_k^s}(\mathbf{x}_1)\Big) \cap \Big(\bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_k^s}(\mathbf{x}_2)\Big). \tag{2.14}$$



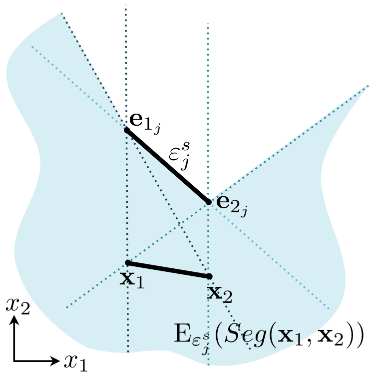
(a) $\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0$.



(b) $\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \wedge \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) > 0$.



(c) $\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) < 0 \wedge \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) < 0$.



(d) $\mathrm{E}_{\varepsilon^s_j}(Seg(\mathbf{x}_1, \mathbf{x}_2))$.

Figure 2.5: Visible space of a segment regards to a segment : case where $\zeta_{x_1} = \zeta_{x_2}$.

Proof

$$
\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon^p_j}(Seg(\mathbf{x}_1, \mathbf{x}_2)) &= \bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon^s_k}(Seg(\mathbf{x}_1, \mathbf{x}_2)) \text{ (Eq. 2.4)},\\
&= \bigcup_{k=1}^{n_{P_j}} (\widehat{\mathrm{E}}_{\varepsilon^s_k}(\mathbf{x}_1) \cap \widehat{\mathrm{E}}_{\varepsilon^s_k}(\mathbf{x}_2)) \text{ (Eq. 2.11)},\\
\widehat{\mathrm{E}}_{\varepsilon^p_j}(Seg(\mathbf{x}_1, \mathbf{x}_2)) &= \Big(\bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon^s_k}(\mathbf{x}_1)\Big) \cap \Big(\bigcup_{k=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon^s_k}(\mathbf{x}_2)\Big).
\end{aligned}
$$

Figure 2.7 illustrates Propositions 2.2.3 and 2.2.4.

2.3 Visibility of a box

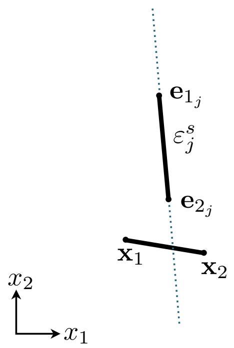
As it is indicated in the appendix, a box $[\mathbf{x}]$ can be associated to four segments, Figure A.5. In the later, $P_{\mathbf{x}}$ denotes the convex polygon associated to the box $[\mathbf{x}] = [x_1] \times [x_2]$ with $[x_1] = [\underline{x_1}, \overline{x_1}]$ and $[x_2] = [\underline{x_2}, \overline{x_2}]$.

$$
\begin{aligned}
P_{\mathbf{x}} &= \text{polygon defined by } \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \text{ and } \mathbf{p}_4,\\
P_{\mathbf{x}} &= \bigcup_{k=1}^{n_P} Seg(\mathbf{p}_k, \mathbf{p}_{k+1}), \text{ with } n_P = 4 \text{ and } \mathbf{p}_5 \equiv \mathbf{p}_1,
\end{aligned}
$$

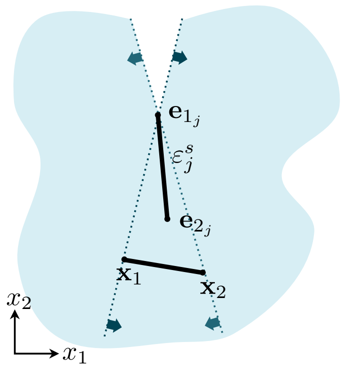
with

$$
\mathbf{p}_1 = (\underline{x_1}, \underline{x_2}), \mathbf{p}_2 = (\overline{x_1}, \underline{x_2}), \mathbf{p}_3 = (\overline{x_1}, \overline{x_2}), \mathbf{p}_4 = (\underline{x_1}, \overline{x_2}). \tag{2.15}
$$

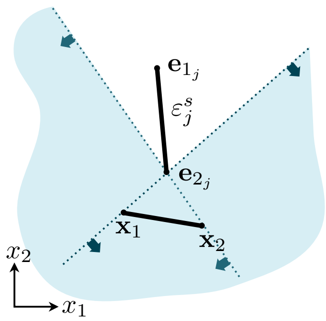
Thus the propositions of the previous section can be applied in order to characterize the visibility spaces of a box.



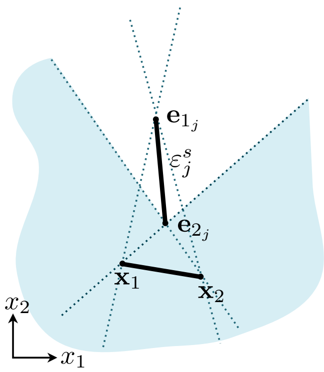
(a) $\zeta_{x_1} = -\zeta_{x_2}$.



(b) $\zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \vee \zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j})$.

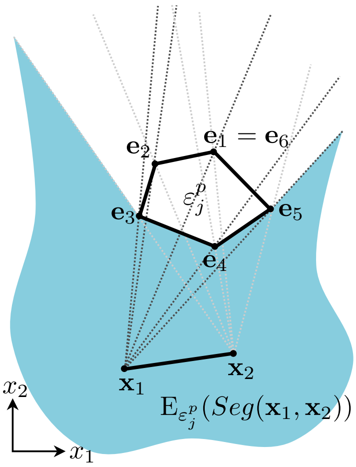


(c) $\zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \vee \zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j})$.

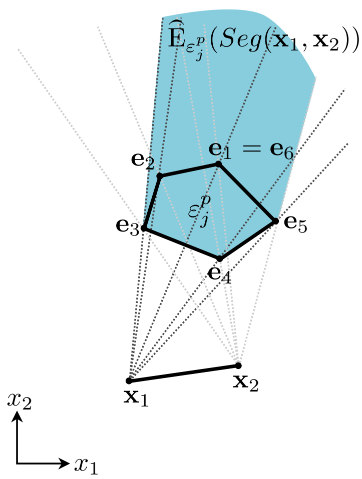


(d) $\mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2))$.

Figure 2.6: Visible space of a segment regards to a segment: case where $\zeta_{x_1} = -\zeta_{x_2}$.

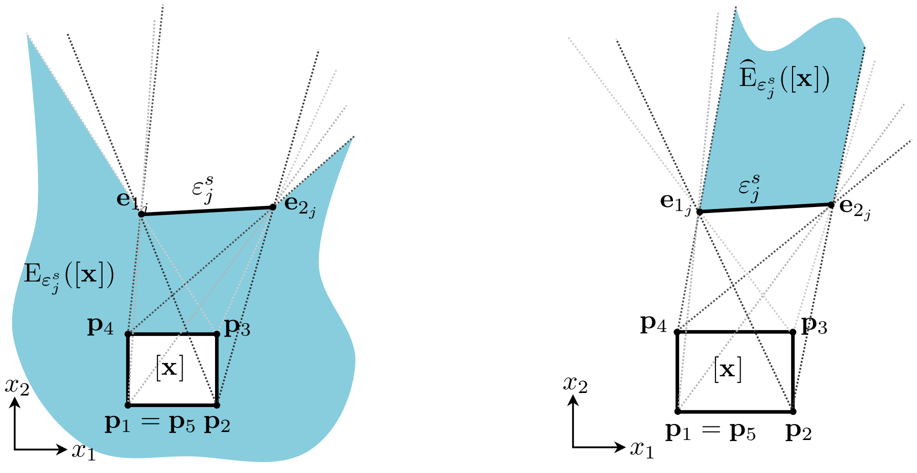


(a) $\mathrm{E}_{\varepsilon_j^p}(Seg(\mathbf{x}_1, \mathbf{x}_2))$.



(b) $\widehat{\mathrm{E}}_{\varepsilon_j^p}(Seg(\mathbf{x}_1, \mathbf{x}_2))$.

Figure 2.7: Visible et non-visible spaces of a segment regards to a polygon.



(a) $\mathrm{E}_{\varepsilon_j^s}([\mathbf{x}]) = \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}})$. (b) $\widehat{\mathrm{E}}_{\varepsilon_j^s}([\mathbf{x}]) = \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}})$.

Figure 2.8: Visible and non-visible spaces of a box regards to a segment.

2.3.1 Regards to a segment obstacle

As denoted previously, the visibility spaces of a box can be characterized by the visibility spaces of the associated segments.

Proposition 2.3.1 *Let* $[\mathbf{x}]$ *be a box with* $P_{\mathbf{x}}$ *its corresponding polygon , and* $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ *an obstacle with* $P_{\mathbf{x}} \cap \varepsilon_j^s = \emptyset$*. Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}([\mathbf{x}]) = \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})). \tag{2.16}$$

Proof

$$\begin{aligned}
\mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) &\Leftrightarrow \forall \mathbf{x}_p \in P_{\mathbf{x}}, (\mathbf{x}_i \overline{\nabla} \mathbf{x}_p)_{\varepsilon_j^s} \text{ (Eq. 1.17)},\\
&\Leftrightarrow \forall \mathbf{x}_p \in P_{\mathbf{x}}, \mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j^s}(\mathbf{x}_p) \text{ (Eq. 1.13)},\\
&\Rightarrow \forall Seg(\mathbf{p}_k, \mathbf{p}_{k+1}) \in P_{\mathbf{x}}, \mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})),\\
\mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) &\Rightarrow \mathbf{x}_i \in \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})).
\end{aligned}$$

Then

$$\widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) \supseteq \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})). \tag{2.17}$$

$$\begin{aligned}
\mathbf{x}_i \in \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) &\Leftrightarrow \forall Seg(\mathbf{p}_k, \mathbf{p}_{k+1}) \in P_{\mathbf{x}}, \forall \mathbf{x}_s \in Seg(\mathbf{p}_k, \mathbf{p}_{k+1}),\\
&\qquad (\mathbf{x}_i \overline{\nabla} \mathbf{x}_s)_{\varepsilon_j^s} \text{ (Eq. 1.17)},\\
&\Rightarrow \forall \mathbf{x}_p \in P_{\mathbf{x}}, (\mathbf{x}_i \overline{\nabla} \mathbf{x}_p)_{\varepsilon_j^s},\\
\mathbf{x}_i \in \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) &\Rightarrow \mathbf{x}_i \in \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) \text{ (Eq. 1.13)}.
\end{aligned}$$

Thus

$$\bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \supseteq \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}). \tag{2.18}$$

It can be concluded

$$\text{(Eq. 2.17 et 2.18)} \Rightarrow \widehat{\mathrm{E}}_{\varepsilon_j^s}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})).$$

And for the non-visible space of a box regards to a segment (Figure 2.8):

Proposition 2.3.2 *Let* $[\mathbf{x}]$ *be a box with* $P_{\mathbf{x}}$ *its corresponding polygon, and* $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_2})$ *an obstacle with* $P_{\mathbf{x}} \cap \varepsilon_j^s = \emptyset$*. Then*

$$\mathrm{E}_{\varepsilon_j^p}([\mathbf{x}]) = \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})). \tag{2.19}$$

Proof

$$\begin{aligned}
\mathbf{x}_i \in \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) \Leftrightarrow & \forall \mathbf{x}_p \in P_{\mathbf{x}}, (\mathbf{x}_i \mathrm{V} \mathbf{x}_p)_{\varepsilon_j^s} \text{ (Eq. 1.16)},\\
\Leftrightarrow & \forall \mathbf{x}_p \in P_{\mathbf{x}}, \mathbf{x}_i \in \mathrm{E}_{\varepsilon_j^s}(\mathbf{x}_p) \text{ (Eq. 1.10)},\\
\Rightarrow & \forall Seg(\mathbf{p}_k, \mathbf{p}_{k+1}) \in P_{\mathbf{x}}, \mathbf{x}_i \in \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})),\\
\mathbf{x}_i \in \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) \Rightarrow & \mathbf{x}_i \in \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})).
\end{aligned}$$

Then

$$\mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) \supseteq \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})). \tag{2.20}$$

$$\begin{aligned}
\mathbf{x}_i \in \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Leftrightarrow & \forall Seg(\mathbf{p}_k, \mathbf{p}_{k+1}) \in P_{\mathbf{x}}, \forall \mathbf{x}_s \in Seg(\mathbf{p}_k, \mathbf{p}_{k+1}),\\
& (\mathbf{x}_i \mathrm{V} \mathbf{x}_s)_{\varepsilon_j^s} \text{ (Eq. 1.16)},\\
\Rightarrow & \forall \mathbf{x}_p \in P_{\mathbf{x}}, (\mathbf{x}_i \mathrm{V} \mathbf{x}_p)_{\varepsilon_j^s},\\
\mathbf{x}_i \in \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Rightarrow & \mathbf{x}_i \in \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) \text{ (Eq. 1.10)}.
\end{aligned}$$

Thus

$$\bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \supseteq \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}). \tag{2.21}$$

It can be concluded

$$\text{(Eq. 2.20 and 2.21)} \Rightarrow \mathrm{E}_{\varepsilon_j^s}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})).$$

The last step consists of characterizing the visibility spaces of a box regards to a polygon obstacle.

2.3.2 Regards to a polygon obstacle

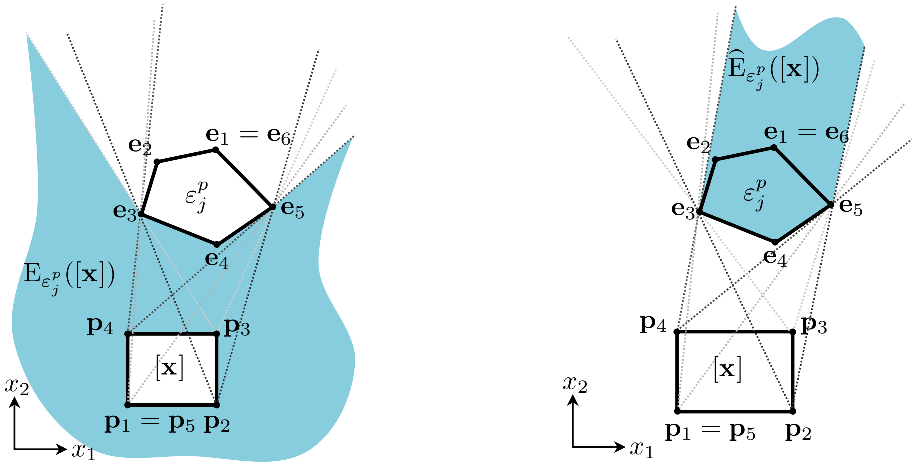
Proposition 2.3.3 *Let* $[\mathbf{x}]$ *be a box with* $P_{\mathbf{x}}$ *its corresponding polygon, and* ε_j^p *an obstacle defined by* n_{P_j} *edges with* $P_{\mathbf{x}} \cap \varepsilon_j^p = \emptyset$*. Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}([\mathbf{x}]) = \mathrm{E}_{\varepsilon_j^p}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \Big(\bigcap_{k'=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_{k'}^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Big). \tag{2.22}$$

Proof

$$\begin{aligned}
\mathrm{E}_{\varepsilon_j^p}(P_{\mathbf{x}}) &= \bigcap_{k'=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_{k'}^s}(P_{\mathbf{x}}) \text{ (Eq. 1.32 et 2.2)},\\
\mathrm{E}_{\varepsilon_j^p}(P_{\mathbf{x}}) &= \bigcap_{k'=1}^{n_{P_j}} \Big(\bigcap_{k=1}^{n_P} \mathrm{E}_{\varepsilon_{k'}^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Big) \text{ (Eq. 2.19)},\\
\mathrm{E}_{\varepsilon_j^p}(P_{\mathbf{x}}) &= \bigcap_{k=1}^{n_P} \Big(\bigcap_{k'=1}^{n_{P_j}} \mathrm{E}_{\varepsilon_{k'}^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Big).
\end{aligned}$$

And for the non-visible space it can be defined:



(a) $\mathrm{E}_{\varepsilon_j^p}([\mathbf{x}]) = \mathrm{E}_{\varepsilon_j^p}(P_{\mathbf{x}})$.

(b) $\widehat{\mathrm{E}}_{\varepsilon_j^p}([\mathbf{x}]) = \widehat{\mathrm{E}}_{\varepsilon_j^p}(P_{\mathbf{x}})$.

Figure 2.9: Visible and non-visible spaces of a box regards to a convex polygon.

Proposition 2.3.4 *Let $[\mathbf{x}]$ be a box with $P_{\mathbf{x}}$ its associated polygon, and ε_j^p an obstacle defined by n_{P_j} edges with $P \cap \varepsilon_j^p = \emptyset$. Then*

$$\widehat{\mathrm{E}}_{\varepsilon_j^p}([\mathbf{x}]) = \widehat{\mathrm{E}}_{\varepsilon_j^p}(P_{\mathbf{x}}) = \bigcap_{k=1}^{n_P} \Big(\bigcup_{k'=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(\mathbf{p}_k) \Big). \tag{2.23}$$

Proof

$$\begin{aligned}
\widehat{\mathrm{E}}_{\varepsilon_j^p}(P_{\mathbf{x}}) &= \bigcup_{k'=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(P_{\mathbf{x}}) \text{ (Eq. 2.4)},\\
&= \bigcup_{k'=1}^{n_{P_j}} \Big(\bigcap_{k}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(Seg(\mathbf{p}_k, \mathbf{p}_{k+1})) \Big) \text{ (Eq. 2.16)},\\
&= \bigcup_{k'=1}^{n_{P_j}} \Big(\bigcap_{k=1}^{n_P} (\widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(\mathbf{p}_k) \cap \widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(\mathbf{p}_{k+1})) \Big) \text{ (Eq. 2.11)},\\
&= \bigcup_{k'=1}^{n_{P_j}} \Big(\bigcap_{k=1}^{n_P} \widehat{\mathrm{E}}_{\varepsilon_k^s}(\mathbf{p}_k) \Big),\\
\widehat{\mathrm{E}}_{\varepsilon_j^p}(P_{\mathbf{x}}) &= \bigcap_{k=1}^{n_P} \Big(\bigcup_{k'=1}^{n_{P_j}} \widehat{\mathrm{E}}_{\varepsilon_{k'}^s}(\mathbf{p}_k) \Big).
\end{aligned}$$

Figure 2.9 illustrates Propositions 2.3.3 and 2.3.4.

We now have all the needed tools to detail the contractors.

Chapter 3

Visibility Contractors

Let $\mathbf{x}_1 \in [\mathbf{x}_1]$ and $\mathbf{x}_2 \in [\mathbf{x}_2]$ be two points, and ε_j^p a polygon obstacle. The idea is to reduce the domains $[\mathbf{x}_1]$ and $[\mathbf{x}_2]$ according to a visibility constraint $(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j^p}$ or a non-visibility constraint $(\mathbf{x}_1 \overline{\mathrm{V}} \mathbf{x}_2)_{\varepsilon_j^p}$ (Figure 3.1).

To be able to contract the domains, by using the propositions presented previously, we define contractors (algorithms). Those contractors rely on the following

Proposition 3.0.5 *Let* $\mathbf{a} \in \mathbb{A}$ *be a point with* $\mathbb{A} \subset \mathbb{R}^n$, $\mathbf{x} \in [\mathbf{x}]$ *be a variable and* $\mathcal{E}$ *an environment defined in* $\mathbb{R}^n$. *Then*

$$(\mathbf{a}\mathrm{V}\mathbf{x})_{\mathcal{E}} \Rightarrow \mathbf{x} \in [\mathbf{x}] \cap \left(\widehat{\mathrm{E}}_{\mathcal{E}}(\mathbb{A})\right)^c \tag{3.1}$$

$$(\mathbf{a}\overline{\mathrm{V}}\mathbf{x})_{\mathcal{E}} \Rightarrow \mathbf{x} \in [\mathbf{x}] \cap \left(\mathrm{E}_{\mathcal{E}}(\mathbb{A})\right)^c \tag{3.2}$$

Proof This proposition is deduced from Proposition 1.3.1.

The basic idea of the contractors is to use the characterizations of the visible and non-visible spaces to contract the domains.

As previously, the contractors are going to be described step by step. First the contractors associated to a point visibility are presented, then those associated to a segment visibility and finally to a box visibility.

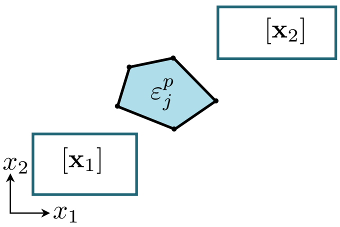
It can be noticed that the contractors C_{det}, $C_{\cap=\emptyset}$, and $C_{\cap\neq\emptyset}$, used in the following, are presented in the appendix.

3.1 Contractions over a point visibility information

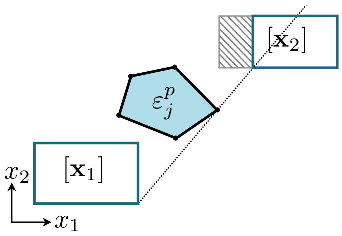
3.1.1 Point visibility contractor regards to a segment obstacle

This contractor, named $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$, is associated to the constraint

$$(\mathbf{x}\mathrm{V}\mathbf{a})_{\varepsilon_j^s}, \tag{3.3}$$

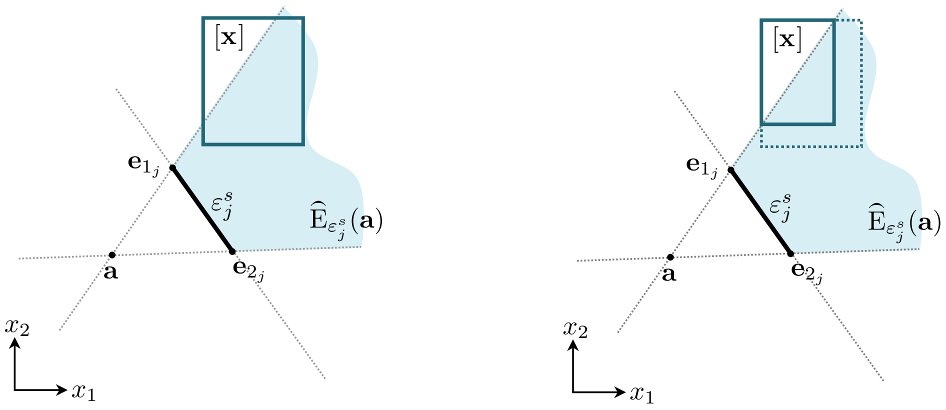


(a) Let $\mathbf{x}_1 \in [\mathbf{x}_1]$ and $\mathbf{x}_2 \in [\mathbf{x}_2]$ be two points, and ε_j^p a polygon obstacle.



(b) Knowing that $(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j^p}$, the objective is to contract (reduce) $[\mathbf{x}_1]$ and $[\mathbf{x}_2]$ by removing the values that are not compatible with the visibility constraint: the hatched area belongs to the non-visible space of $[\mathbf{x}_1]$ regards to ε_j^p, this area is then not feasible with the visibility constraint and can be *removed* from the domain $[\mathbf{x}_2]$.

Figure 3.1: Contraction example according to the constraint $(\mathbf{x}_1 \mathrm{V} \mathbf{x}_2)_{\varepsilon_j^p}$.



(a) $(\mathbf{x}\mathrm{V}\mathbf{a})_{\varepsilon_j^s}$, with $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.2: Presentation of the contractor $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$.

with $\mathbf{a} \in \mathbb{R}^2$, $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$ and $\mathbf{x} \in [\mathbf{x}]$. Figure 3.2 illustrates its principle.

This contractor is presented Algorithm 1. It is based on the complement of Equation 2.8.

Algorithm 1: Contractor $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$

Input: $[\mathbf{x}], \mathbf{a}, \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$

if $\det(\mathbf{a} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0$ **then**

  $\zeta_a = 1$;

else

  $\zeta_a = -1$;

$[\mathbf{i}_1] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{e}_{2_j}, \zeta_a)$;

$[\mathbf{i}_2] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{a}, \zeta_a)$;

$[\mathbf{i}_3] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{a}, -\zeta_a)$;

$[\mathbf{i}_4] = C_{\cap=\emptyset}([\mathbf{x}], \mathbf{a}, \mathbf{e}_{1_j}, \mathbf{e}_2)$;

Output: $[\mathbf{x}]^* = [\mathbf{i}_1] \cup [\mathbf{i}_2] \cup [\mathbf{i}_3] \cup [\mathbf{i}_4]$

3.1.2 Point non-visibility contractor regards to a segment obstacle

The dual of the previous contractor, named $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$, is associated to the constraint

$$(\mathbf{x}\overline{\mathrm{V}}\mathbf{a})_{\varepsilon_j^s}, \tag{3.4}$$

with $\mathbf{a} \in \mathbb{R}^2$, $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$ and $\mathbf{x} \in [\mathbf{x}]$.

This contractor is presented Algorithm 2. It is based on the complement of Equation 2.5. Figure 3.3 illustrates its principle.

Algorithm 2: Contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$

Input: $[\mathbf{x}], \mathbf{a}, \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$

if $\det(\mathbf{a} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0$ **then**

  $\zeta_a = 1$;

else

  $\zeta_a = -1$;

$[\mathbf{i}_1] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{e}_{2_j}, -\zeta_a)$;

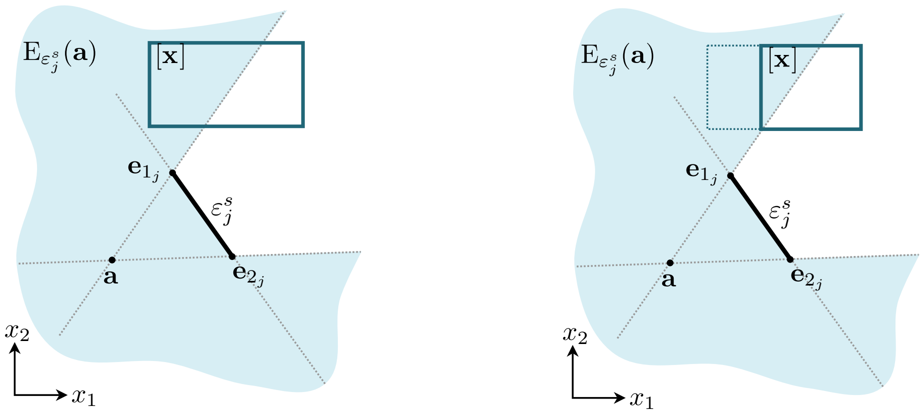
$[\mathbf{i}_2] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{a}, -\zeta_a)$;

$[\mathbf{i}_3] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{a}, \zeta_a)$;

$[\mathbf{i}_4] = C_{\cap\neq\emptyset}([\mathbf{x}], \mathbf{a}, \mathbf{e}_{1_j}, \mathbf{e}_2)$;

Output: $[\mathbf{x}]^* = [\mathbf{i}_1] \cap [\mathbf{i}_2] \cap [\mathbf{i}_3] \cap [\mathbf{i}_4]$

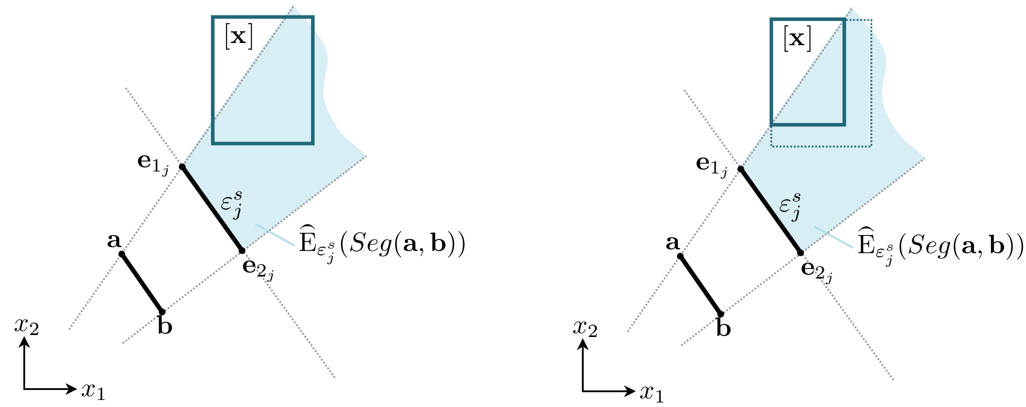
Once those two contractors defined, it is possible de get interested into the contractors associated to a segment visibility information.



(a) $(\mathbf{x}\overline{\mathrm{V}}\mathbf{a})_{\varepsilon_j^s}$, with $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.3: Presentation of the contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$.



(a) $(\mathbf{x}\mathrm{V}\mathbf{x}_{ab})_{\varepsilon_j^s}$, with $\mathbf{x}_{ab} \in Seg(\mathbf{a}, \mathbf{b})$ and $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.4: Presentation of the contractor $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$.

3.2 Contractions over a segment visibility information

3.2.1 Segment visibility contractor regards to a segment obstacle

This contractor, named $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$, is associated to the constraint

$$(\mathbf{x}\mathrm{V}\mathbf{x}_{ab})_{\varepsilon_j^s}, \tag{3.5}$$

with $\mathbf{x}_{ab} \in Seg(\mathbf{a}, \mathbf{b})$, $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^2$, $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$ and $\mathbf{x} \in [\mathbf{x}]$.

It is detailed Algorithm 3. It is based on the complement of Equation 2.11. Figure 3.4 illustrates its principle.

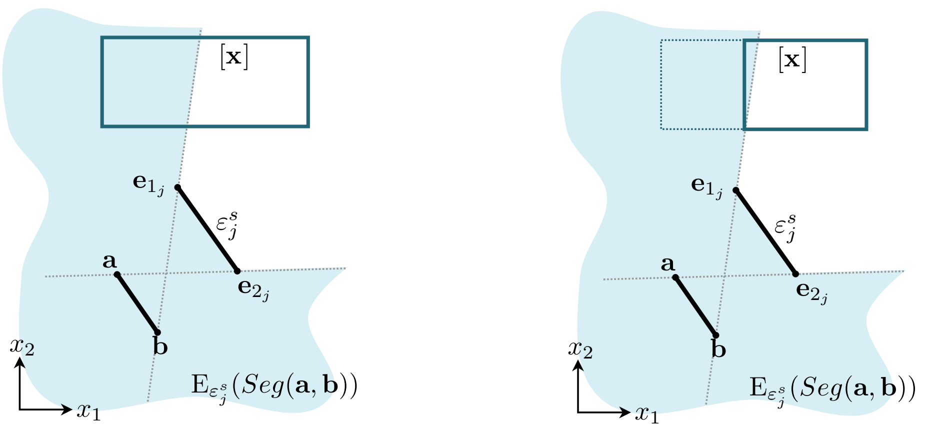
Algorithm 3: Contractor $C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$

Input: $[\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$
$[\mathbf{i}_1] = C_{\mathrm{V}}([\mathbf{x}], \mathbf{a}, \varepsilon_j^s)$;
$[\mathbf{i}_2] = C_{\mathrm{V}}([\mathbf{x}], \mathbf{b}, \varepsilon_j^s)$;
Output: $[\mathbf{x}]^* = [\mathbf{i}_1] \cup [\mathbf{i}_2]$

3.2.2 Segment non-visibility contractor regards to a segment obstacle

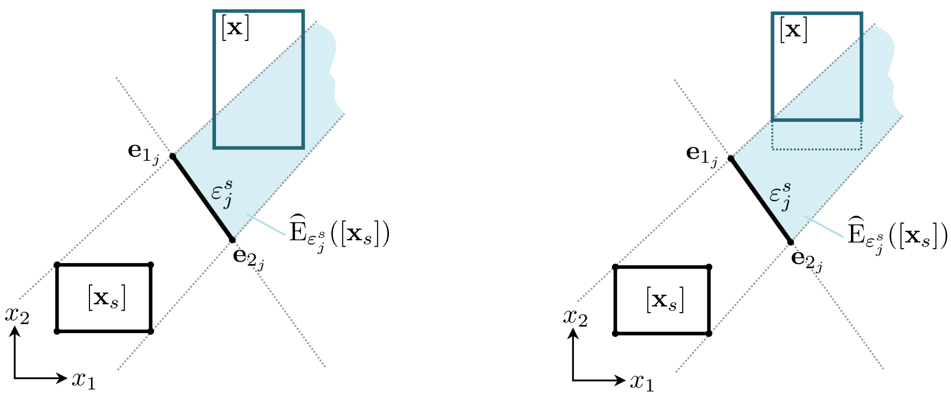
The dual of the previous contractor, named $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$, is associated to the constraint

$$(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_{ab})_{\varepsilon_j^s}, \tag{3.6}$$



(a) $(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_{ab})_{\varepsilon_j^s}$, with $\mathbf{x}_{ab} \in Seg(\mathbf{a},\mathbf{b})$ and $\mathbf{x} \in [\mathbf{x}]$. (b) Contraction of $[\mathbf{x}]$.

Figure 3.5: Presentation of the contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$.



(a) $(\mathbf{x}\mathrm{V}\mathbf{x}_s)_{\varepsilon_j^s}$, with $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$. (b) Contraction of $[\mathbf{x}]$.

Figure 3.6: Presentation of the contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$.

with $\mathbf{x}_{ab} \in Seg(\mathbf{a},\mathbf{b})$, $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^2$, $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$ and $\mathbf{x} \in [\mathbf{x}]$.

This contractor is presented Algorithm 4. It is based on the complement of Equation 2.12. Figure 3.5 illustrates its principle.

3.3 Contractions over a box visibility information

3.3.1 Box visibility contractor regards to a segment obstacle

The contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$ is associated to the constraint

$$(\mathbf{x}\mathrm{V}\mathbf{x}_s)_{\varepsilon_j^s}, \tag{3.7}$$

with $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$, $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$. Figure 3.6 illustrates its principle. This contractor is presented Algorithm 5. It is based on the complement of Equation 2.16.

3.3.2 Box non-visibility contractor regards to a segment obstacle

The contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$ is associated to the constraint

$$(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_s)_{\varepsilon_j^s}, \tag{3.8}$$

with $\varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$, $\mathbf{e}_{1_j} \in \mathbb{R}^2$, $\mathbf{e}_{2_j} \in \mathbb{R}^2$, $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$.

This contractor is presented Algorithm 6. It is based on the complement of Equation 2.19. Figure 3.7 illustrates its principle.

Algorithm 4: Contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s)$

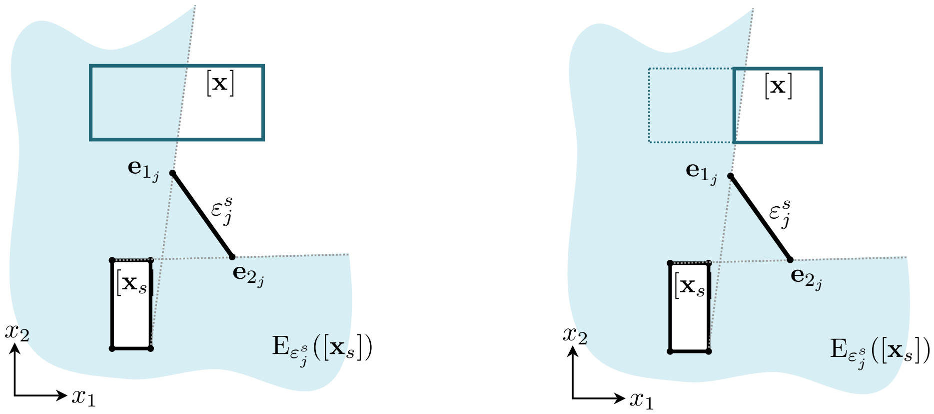
Input: $[\mathbf{x}], \mathbf{a}, \mathbf{b}, \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$
if $\det(\mathbf{a} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0$ **then**
  $\zeta_a = 1$;
else
  $\zeta_a = -1$;
if $\det(\mathbf{b} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0$ **then**
  $\zeta_b = 1$;
else
  $\zeta_b = -1$;
if $\det(\mathbf{e}_{1_j} - \mathbf{a} | \mathbf{b} - \mathbf{a}) > 0$ **then**
  $\zeta_{\mathbf{e}_1} = 1$;
else
  $\zeta_{\mathbf{e}_1} = -1$;
if $\det(\mathbf{e}_{2_j} - \mathbf{a} | \mathbf{b} - \mathbf{a}) > 0$ **then**
  $\zeta_{\mathbf{e}_2} = 1$;
else
  $\zeta_{\mathbf{e}_2} = -1$;
// Two possible cases
if $\zeta_a = \zeta_b$ **then**
  $[\mathbf{i}_{11}] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{e}_{2_j}, -\zeta_a)$;
  $[\mathbf{i}_{12}] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{a}, -\zeta_a)$;
  $[\mathbf{i}_{13}] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{b}, -\zeta_b)$;
  $[\mathbf{i}_{14}] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{a}, \zeta_a)$;
  $[\mathbf{i}_{15}] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{b}, \zeta_b)$;
  $[\mathbf{i}_{output}] = [\mathbf{i}_{11}] \cap ([\mathbf{i}_{12}] \cup [\mathbf{i}_{13}]) \cap ([\mathbf{i}_{14}] \cup [\mathbf{i}_{15}])$
else
  // $\zeta_a = -\zeta_b$
  $[\mathbf{i}_{21}] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{a}, -\zeta_{e_1})$;
  $[\mathbf{i}_{22}] = C_{det}([\mathbf{x}], \mathbf{e}_{1_j}, \mathbf{b}, \zeta_{e_1})$;
  $[\mathbf{i}_{23}] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{a}, -\zeta_{e_2})$;
  $[\mathbf{i}_{24}] = C_{det}([\mathbf{x}], \mathbf{e}_{2_j}, \mathbf{b}, \zeta_{e_2})$;
  $[\mathbf{i}_{output}] = ([\mathbf{i}_{21}] \cap [\mathbf{i}_{22}]) \cup ([\mathbf{i}_{23}] \cap [\mathbf{i}_{24}])$
$[\mathbf{i}_0] = C_{\cap \neq \emptyset}([\mathbf{x}], \mathbf{a}, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) \cup C_{\cap \neq \emptyset}([\mathbf{x}], \mathbf{b}, \mathbf{e}_{1_j}, \mathbf{e}_{2_j})$;
Output: $[\mathbf{x}]^* = [\mathbf{i}_{output}] \cap [\mathbf{i}_0]$

Algorithm 5: Contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$

Input: $[\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$
P_s = polygon associated to $[\mathbf{x}_s]$;
for all $\mathbf{p}_k \in P_s$ **do**
  $[\mathbf{i}_k] = C_{\mathrm{V}}([\mathbf{x}], \mathbf{p}_k, \mathbf{p}_{k+1}, \varepsilon_j^s)$;
Output: $[\mathbf{x}]^* = \bigcup_{k=1}^{n_P} [\mathbf{i}_k]$

Algorithm 6: Contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$

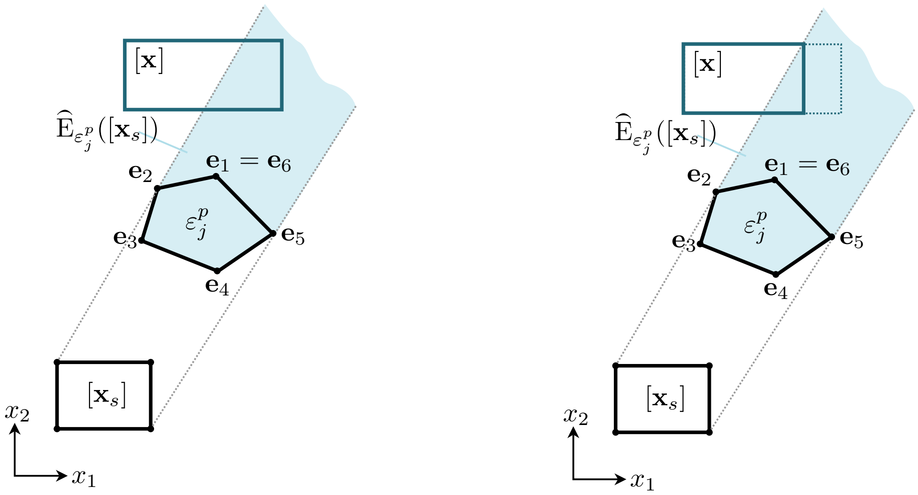
Input: $[\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s = Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$
P_s = polygon associated to $[\mathbf{x}_s]$;
for all $\mathbf{p}_k \in P_s$ **do**
  $[\mathbf{i}_k] = C_{\overline{\mathrm{V}}}([\mathbf{x}], \mathbf{p}_k, \mathbf{p}_{k+1}, \varepsilon_j^s)$;
Output: $[\mathbf{x}]^* = \bigcup_{k=1}^{n_P} [\mathbf{i}_k]$



(a) $(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_s)_{\varepsilon_j^s}$, with $\mathbf{x}_s \in [\mathbf{x}_s]$ et $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.7: Presentation of the contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^s)$.



(a) $(\mathbf{x}\mathrm{V}\mathbf{x}_s)_{\varepsilon_j^p}$, with $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.8: Presentation of the contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$.

3.3.3 Box visibility contractor regards to a polygon obstacle

The contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$ is associated to the constraint

$$(\mathbf{x}\mathrm{V}\mathbf{x}_s)_{\varepsilon_j^p}, \tag{3.9}$$

with ε_j^p a polygon defined by n_{P_j} edges $\mathbf{e}_k \in \mathbb{R}^2$, $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$.

This contractor is presented Algorithm 7. It is based on the complement of Equation 2.23. Figure 3.8 illustrates its principle.

3.3.4 Box non-visibility contractor regards to a polygon obstacle

The contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$ is associated to the constraint

$$(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_s)_{\varepsilon_j^p}, \tag{3.10}$$

with ε_j^p a polygon defined by n_{P_j} edges $\mathbf{e}_k \in \mathbb{R}^2$, $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$.

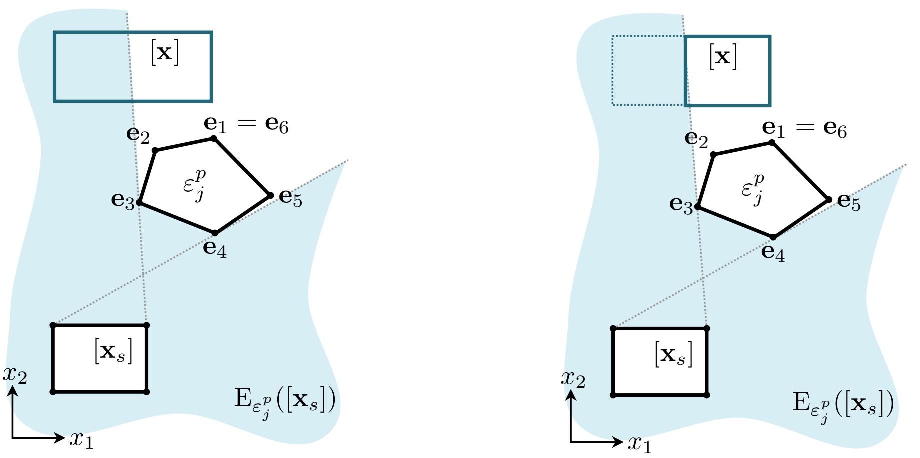
This contractor is presented Algorithm 8. It is based on the complement of Equation 2.22. Figure 3.9 illustrates its principle.

Algorithm 7: Contractor $C_{\mathrm{V}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$

Input: $[\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p$

P_s = polygon associated to $[\mathbf{x}_s]$;
for all $\mathbf{p}_k \in P_s$ **do**
  for all $\mathbf{e}_{k'} \in \varepsilon_j^p$ **do**
    $[\mathbf{j}_{k'}] = C_{\mathrm{V}}([\mathbf{x}], \mathbf{p}_k, Seg(\mathbf{e}_{k'}, \mathbf{e}_{k'+1}))$;
  $[\mathbf{i}_k] = \bigcap_{k'=1}^{n_{P_j}} [\mathbf{j}_{k'}]$;

Output: $[\mathbf{x}]^* = \bigcup_{k=1}^{n_P} [\mathbf{i}_k]$



(a) $(\mathbf{x}\overline{\mathrm{V}}\mathbf{x}_s)_{\varepsilon_j^p}$, with $\mathbf{x}_s \in [\mathbf{x}_s]$ and $\mathbf{x} \in [\mathbf{x}]$.

(b) Contraction of $[\mathbf{x}]$.

Figure 3.9: Presentation of the contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$.

Algorithm 8: Contractor $C_{\overline{\mathrm{V}}}([\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p)$

Input: $[\mathbf{x}], [\mathbf{x}_s], \varepsilon_j^p$

P_s = polygon associated to $[\mathbf{x}_s]$;
for all $\mathbf{p}_k \in P_s$ **do**
  for all $\mathbf{e}_{k'} \in \varepsilon_j^p$ **do**
    $[\mathbf{j}_{k'}] = C_{\overline{\mathrm{V}}}([x], \mathbf{p}_k, \mathbf{p}_{k+1}, Seg(\mathbf{e}_{k'}, \mathbf{e}_{k'+1}))$;
  $[\mathbf{i}_k] = \bigcup_{k'=1}^{n_{P_j}} [\mathbf{j}_{k'}]$;

Output: $[\mathbf{x}]^* = \bigcup_{k=1}^{n_P} [\mathbf{i}_k]$

Bibliography

[Jaulin 2001a] L. Jaulin, M. Kieffer, O. Didrit et E. Walter. Applied interval analysis. Springer, 2001.

[Jaulin 2001b] Luc Jaulin. *Path Planning Using Intervals and Graphs.* Reliable Computing, vol. 7, no. 1, pages 1–15, 2001.

Appendix A

Segment Tools

A.1 Parametric Equation of a Segment

Let $Seg(\mathbf{a}, \mathbf{b})$ be a segment with $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. Every point $\mathbf{s} = (s_1, s_2) \in Seg(\mathbf{a}, \mathbf{b})$ can be written as

$$\begin{cases} s_1 = (1-t)a_1 + tb_1, \\ s_2 = (1-t)a_2 + tb_2, \\ t \in [0, 1]. \end{cases} \tag{A.1}$$

This notation corresponds to the parametric equation of the segment $Seg(\mathbf{a}, \mathbf{b})$.

A.2 Segment Intersection

A.2.1 Point Position Regards to a Segment

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ and $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ be three points,

$$\begin{aligned} \det(\mathbf{a} - \mathbf{b} | \mathbf{c} - \mathbf{b}) &= \det \begin{pmatrix} a_1 - b_1 & c_1 - b_1 \\ a_2 - b_2 & c_2 - b_2 \end{pmatrix}, \\ \Rightarrow \quad \det(\mathbf{a} - \mathbf{b} | \mathbf{c} - \mathbf{b}) &= (a_1 - b_1)(c_2 - b_2) - (a_2 - b_2)(c_1 - b_1). \end{aligned} \tag{A.2}$$

The sign of $\det(\mathbf{a} - \mathbf{b} | \mathbf{c} - \mathbf{b})$ characterizes the *position* of the point $\mathbf{a}$ regards to the line associated to the vector $\overrightarrow{\mathbf{bc}}$. Figure A.1 illustrates the different cases.

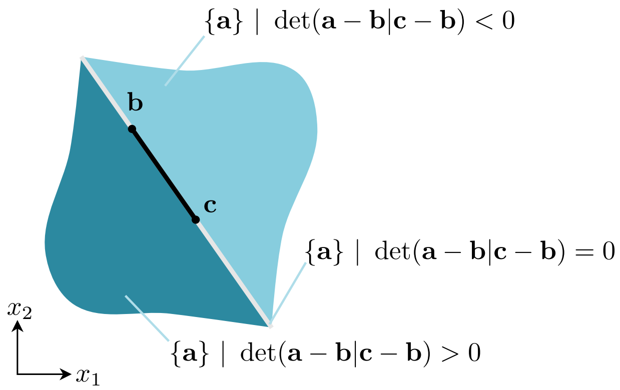


Figure A.1: Different cases for the sign of $\det(\mathbf{a} - \mathbf{b} | \mathbf{c} - \mathbf{b})$.

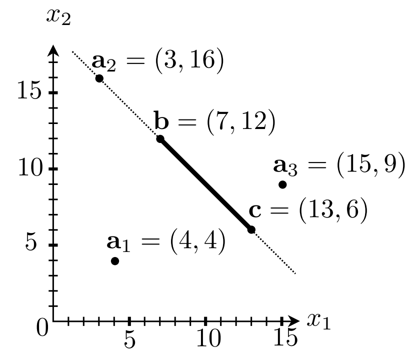


Figure A.2: Case study.

Considering the example in Figure A.2:

$$
\begin{aligned}
\det(\mathbf{a}_1 - \mathbf{b}|\mathbf{c} - \mathbf{b}) &= (4-7)(6-12) - (4-12)(13-7),\\
&= 66 > 0.\\
\det(\mathbf{a}_2 - \mathbf{b}|\mathbf{c} - \mathbf{b}) &= (3-7)(6-12) - (16-12)(13-7),\\
&= 0.\\
\det(\mathbf{a}_3 - \mathbf{b}|\mathbf{c} - \mathbf{b}) &= (15-7)(6-12) - (9-12)(13-7),\\
&= -30 < 0.
\end{aligned}
$$

Algorithme 9 presents the contractor associated to the constraint

$$\zeta \det(\mathbf{x} - \mathbf{a}|\mathbf{b} - \mathbf{a}) \geq 0 \tag{A.3}$$

with $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ two known points and $\zeta = \{-1, 1\}$. By considering Equation A.3

$$x_1\zeta(b_2 - a_2) - x_2\zeta(b_1 - a_1) - \zeta\Big(a_2(b_1 - a_1) - a_1(b_2 - a_2)\Big) \geq 0 \tag{A.4}$$

Algorithm 9: Contractor $C_{det}([\mathbf{x}], \mathbf{a}, \mathbf{b}, \zeta)$

Input: $[\mathbf{x}] = ([x_1], [x_2]), \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2), \zeta$

// Temporary variable to facilitate the reading
$cst_1 = \zeta(b_1 - a_1)$, $cst_2 = \zeta(b_2 - a_2)$, $cst_3 = \zeta\Big(a_2(b_1 - a_1) - a_1(b_2 - a_2)\Big)$;
// Initialization
$[i_1] = [x_1]cst_2$, $[i_2] = [x_2]cst_1$, $[i_3] = [i_1] - [i_2] + cst_3$;
// Contractions
$[i_3]^* = [i_3] \cap \mathbb{R}^+$;
$[i_1]^* = [i_1] \cap ([i_3]^* - cst_3 + [i_2])$;
$[i_2]^* = [i_2] \cap ([i_1]^* + cst_3 - [i_3]^*)$;
if $cst2 \neq 0$ **then**
  $[x_1]^* = [x_1] \cap ([i_1]^*/cst_2)$;
else
  $[x_1]^* = [x_1]$;
if $cst1 \neq 0$ **then**
  $[x_2]^* = [x_2] \cap ([i_2]^*/cst_1)$;
else
  $[x_2]^* = [x_2]$;

Output: $[\mathbf{x}]^* = ([x_1]^*, [x_2]^*)$.

A.2.2 Intersection Test

The general idea to test the intersection between two segments is to test if the points of each segment are on both part of the other one [Jaulin 2001a].

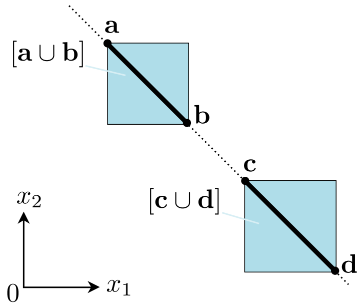


Figure A.3: Particular case for the intersection: the two segments are on the same line. In this case $\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\cdot\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})=0$ and $\det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a})\cdot\det(\mathbf{d}-\mathbf{a}|\mathbf{b}-\mathbf{a})=0$ but the two segments do not intersect each other.

Be four distinct points of $\mathbb{R}^2$, $\mathbf{a}=(a_1,a_2)$, $\mathbf{b}=(b_1,b_2)$, $\mathbf{c}=(c_1,c_2)$ and $\mathbf{d}=(d_1,d_2)$. Then

$$\begin{aligned}
Seg(\mathbf{a},\mathbf{b})\cap Seg(\mathbf{c},\mathbf{d})=\emptyset \Leftrightarrow\, & \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\cdot\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})>0\,\vee\\
& \det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a})\cdot\det(\mathbf{d}-\mathbf{a}|\mathbf{b}-\mathbf{a})>0\,\vee\\
& [\mathbf{a}\cup\mathbf{b}]\cap[\mathbf{c}\cup\mathbf{d}]=\emptyset, \qquad (A.5)\\
Seg(\mathbf{a},\mathbf{b})\cap Seg(\mathbf{c},\mathbf{d})\neq\emptyset \Leftrightarrow\, & \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\cdot\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})\leq 0\wedge\\
& \det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a})\cdot\det(\mathbf{d}-\mathbf{a}|\mathbf{b}-\mathbf{a})\leq 0\,\wedge\\
& [\mathbf{a}\cup\mathbf{b}]\cap[\mathbf{c}\cup\mathbf{d}]\neq\emptyset. \qquad (A.6)
\end{aligned}$$

It can be noticed that it is needed to test $[\mathbf{a}\cup\mathbf{b}]\cap[\mathbf{c}\cup\mathbf{d}]$ to avoid the situation depicted in Figure A.3.

A.2.3 Contractor for $[\mathbf{a}\cup\mathbf{x}]\cap[\mathbf{c}\cup\mathbf{d}]$

We want the contractor associated to the constraint

$$[\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]\neq\emptyset, \qquad (A.7)$$

with $\mathbf{a}\in\mathbb{R}^2$, $\mathbf{b}\in\mathbb{R}^2$ and $\mathbf{c}\in\mathbb{R}^2$ three known points and $[\mathbf{x}]\in\mathbb{IR}^2$.

By considering Equation A.7:

$$\begin{aligned}
& [\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]\neq\emptyset\\
&\quad\Leftrightarrow [\min(\mathbf{a},[\mathbf{x}]),\max(\mathbf{a},[\mathbf{x}])]\cap[\min(\mathbf{c},\mathbf{d}),\max(\mathbf{c},\mathbf{d})]\neq\emptyset\\
&\quad\Leftrightarrow [\max(\min(\mathbf{a},[\mathbf{x}]),\min(\mathbf{c},\mathbf{d})),\min(\max(\mathbf{a},[\mathbf{x}]),\max(\mathbf{c},\mathbf{d}))]\neq\emptyset\\
&\quad\Leftrightarrow \max(\min(\mathbf{a},[\mathbf{x}]),\min(\mathbf{c},\mathbf{d}))\leq\min(\max(\mathbf{a},[\mathbf{x}]),\max(\mathbf{c},\mathbf{d}))\\
&\quad\Leftrightarrow \max(\min(\mathbf{a},[\mathbf{x}]),\min(\mathbf{c},\mathbf{d}))-\min(\max(\mathbf{a},[\mathbf{x}]),\max(\mathbf{c},\mathbf{d}))\leq 0\\
&\quad\Leftrightarrow \begin{pmatrix}\max(\min(a_1,[x_1]),\min(c_1,d_1))-\min(\max(a_1,[x_1]),\max(c_1,d_1))\\ \max(\min(a_2,[x_2]),\min(c_2,d_2))-\min(\max(a_2,[x_2]),\max(c_2,d_2))\end{pmatrix}\leq\begin{pmatrix}0\\0\end{pmatrix}\\
&\quad\Leftrightarrow (\max(\min(a_1,[x_1]),\min(c_1,d_1))-\min(\max(a_1,[x_1]),\max(c_1,d_1)))\leq 0\,\wedge\\
&\qquad\quad (\max(\min(a_2,[x_2]),\min(c_2,d_2))-\min(\max(a_2,[x_2]),\max(c_2,d_2)))\leq 0 \qquad (A.8)
\end{aligned}$$

with

$$\min(a,[x])=\min([x],a)=[\min(a,\underline{x}),\min(a,\overline{x})], \qquad (A.9)$$
$$\max(a,[x])=\max([x],a)=[\max(a,\underline{x}),\max(a,\overline{x})]. \qquad (A.10)$$

It is needed to have contractors associated to the operators min () and max (). Algorithm 10 corresponds to the contractor associated to the constraint

$$[y]=\min([x],a), \qquad (A.11)$$

and Algorithm 11 corresponds to the contractor associated to the constraint

$$[y]=\max([x],a). \qquad (A.12)$$

The two previous algorithms are inspired from [Jaulin 2001b].

Algorithm 10: $C_{min}([y], [x], a)$

Input: $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], a$

// Contraction of $[y]$

$[y]^* = [y] \cap \min([x], a)$;

// Contraction of $[x]$

if $a \notin [y]^*$ **then**

  $[x]^* = [x] \cap [y]^*$;

else

  $[x]^* = [x] \cap [\underline{y}^*, +\infty]$;

Output: $[x]^*, [y]^*$.

Algorithm 11: $C_{max}([y], [x], a)$

Input: $[x] = [\underline{x}, \overline{x}], [y] = [\underline{y}, \overline{y}], a$

// Contraction of $[y]$

$[y]^* = [y] \cap \max([x], a)$;

// Contraction of $[x]$

if $a \notin [y]^*$ **then**

  $[x]^* = [x] \cap [y]^*$;

else

  $[x]^* = [x] \cap [-\infty, \overline{y}^*]$;

Output: $[x]^*, [y]^*$.

Algorithm 12: $C_{\cap \neq \emptyset}([x], a, c, d)$

Input: $[x] = [\underline{x}, \overline{x}], a, c, d$

// Initialization

$[i_1] = \min(a, [x])$;

$[i_2] = \max([i_1], \min(c, d))$;

$[i_3] = \max(a, [x])$;

$[i_4] = \min([i_3], \max(c, d))$;

$[i_5] = [i_2] - [i_4]$;

// Contractions

$[i_5]^* = [i_5] \cap [-\infty, 0]$;

$[i_2]^* = [i_5]^* + [i_4]$;

$[i_4]^* = [i_2]^* - [i_5]$;

$([i_3]^*, [i_4]^*) = C_{min}([i_4]^*, [i_3], \max(c, d))$;

$([x]^*, [i_3]^*) = C_{max}([i_3]^*, [x], a)$;

$([i_1]^*, [i_2]^*) = C_{max}([i_2]^*, [i_1], \min(c, d))$;

$([x]^*, [i_1]^*) = C_{min}([i_1]^*, [x]^*, a)$;

Output: $[x]$.

By using those two contractors it becomes possible to develop a contractor associated to the constraint

$$\max\left(\min\left(a,[x]\right),\min\left(c,d\right)\right)-\min\left(\max\left(a,[x]\right),\max\left(c,d\right)\right)\leq 0. \tag{A.13}$$

This new contractor is presented Algorithm 12.

Considering Algorithm 12 and Equation A.8, it is possible to develop the contractor (Algorithme 13) associated to the constraint

$$[\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]\neq\emptyset \tag{A.14}$$

Algorithm 13: $C_{\cap\neq\emptyset}([\mathbf{x}],\mathbf{a},\mathbf{c},\mathbf{d})$

Input: $[\mathbf{x}]=([x_1],[x_2]),\mathbf{a}=(a_1,a_2),\mathbf{c}=(c_1,c_2),\mathbf{d}=(d_1,d_2)$

// Contraction over each component

$[x_1]^*=C_{\cap\neq\emptyset}([x_1],a_1,c_1,d_1)$;

$[x_2]^*=C_{\cap\neq\emptyset}([x_2],a_2,c_2,d_2)$;

Output: $[\mathbf{x}]^*=([x_1]^*,[x_2]^*)$.

We now want to develop the contractor associated to the constraint

$$[\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]=\emptyset, \tag{A.15}$$

with $\mathbf{a}\in\mathbb{R}^2$, $\mathbf{b}\in\mathbb{R}^2$ and $\mathbf{c}\in\mathbb{R}^2$ three known points and $[\mathbf{x}]\in\mathbb{IR}^2$.

Considering Equation A.8:

$$\begin{aligned}&[\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]=\emptyset\\ &\quad\Leftrightarrow\left(\max\left(\min\left(a_1,[x_1]\right),\min\left(c_1,d_1\right)\right)-\min\left(\max\left(a_1,[x_1]\right),\max\left(c_1,d_1\right)\right)\right)>0\ \wedge\\ &\qquad\left(\max\left(\min\left(a_2,[x_2]\right),\min\left(c_2,d_2\right)\right)-\min\left(\max\left(a_2,[x_2]\right),\max\left(c_2,d_2\right)\right)\right)>0\end{aligned} \tag{A.16}$$

Which leads to the contractor (Algorithm 14) associated to the constraint

$$\max\left(\min\left(a,[x]\right),\min\left(c,d\right)\right)-\min\left(\max\left(a,[x]\right),\max\left(c,d\right)\right)>0, \tag{A.17}$$

and the contractor (Algorithm 15) associated to the constraint

$$[\mathbf{a}\cup[\mathbf{x}]]\cap[\mathbf{c}\cup\mathbf{d}]=\emptyset. \tag{A.18}$$

Algorithm 14: $C_{\cap=\emptyset}([x],a,c,d)$

Input: $[x]=[\underline{x},\overline{x}],a,c,d$

// Initialization

$[i_1]=\min\left(a,[x]\right)$;

$[i_2]=\max\left([i_1],\min\left(c,d\right)\right)$;

$[i_3]=\max\left(a,[x]\right)$;

$[i_4]=\min\left([i_3],\max\left(c,d\right)\right)$;

$[i_5]=[i_2]-[i_4]$;

// Contractions

$[i_5]^*=[i_5]\cap[0,+\infty]$;

$[i_2]^*=[i_5]^*+[i_4]$;

$[i_4]^*=[i_2]^*-[i_5]$;

$([i_3]^*,[i_4]^*)=C_{min}([i_4]^*,[i_3],\max\left(c,d\right))$;

$([x]^*,[i_3]^*)=C_{max}([i_3]^*,[x],a)$;

$([i_1]^*,[i_2]^*)=C_{max}([i_2]^*,[i_1],\min\left(c,d\right))$;

$([x]^*,[i_1]^*)=C_{min}([i_1]^*,[x]^*,a)$;

Output: $[x]$.

Remark A.2.1

$$\Big[[\mathbf{x}]\cup\mathbf{a}\cup\mathbf{b}\Big]\cap[\mathbf{c}\cup\mathbf{d}]=\Big[[\mathbf{x}]\cup\mathbf{a}\Big]\cap[\mathbf{c}\cup\mathbf{d}]\cup\Big[[\mathbf{x}]\cup\mathbf{b}\Big]\cap[\mathbf{c}\cup\mathbf{d}]. \tag{A.19}$$

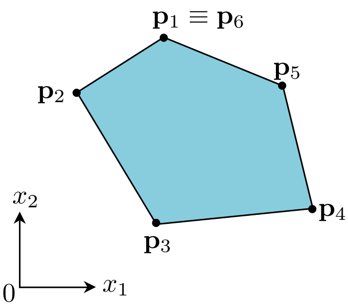


Figure A.4: Polygon P composed by $n_P = 5$ edges.

Proof

$$
\begin{aligned}
&\Big[[\mathbf{x}] \cup \mathbf{a} \cup \mathbf{b}\Big] \cap [\mathbf{c} \cup \mathbf{d}] \\
&\qquad = [\min([\mathbf{x}], \mathbf{a}, \mathbf{b}), \max([\mathbf{x}], \mathbf{a}, \mathbf{b})] \cap [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})].
\end{aligned} \tag{A.20}
$$

$$
\begin{aligned}
&\Big[[\mathbf{x}] \cup \mathbf{a}\Big] \cap [\mathbf{c} \cup \mathbf{d}] \cup \Big[[\mathbf{x}] \cup \mathbf{b}\Big] \cap [\mathbf{c} \cup \mathbf{d}], \\
&\qquad = [\min([\mathbf{x}, \mathbf{a}]), \max([\mathbf{x}, \mathbf{a}])] \cap [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})] \cup \\
&\qquad\qquad [\min([\mathbf{x}], \mathbf{b}), \max([\mathbf{x}], \mathbf{b})] \cap [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})], \\
&\qquad = [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})] \cap \\
&\qquad\qquad \Big([\min([\mathbf{x}], \mathbf{a}), \max([\mathbf{x}], \mathbf{a})] \cup [\min([\mathbf{x}], \mathbf{b}), \max([\mathbf{x}], \mathbf{b})]\Big), \\
&\qquad = [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})] \cap \\
&\qquad\qquad [\min(\min([\mathbf{x}], \mathbf{b}), \min([\mathbf{x}], \mathbf{a})), \max(\max([\mathbf{x}], \mathbf{b}), \max([\mathbf{x}], \mathbf{a}))], \\
&\qquad = [\min(\mathbf{c}, \mathbf{d}), \max(\mathbf{c}, \mathbf{d})] \cap [\min([\mathbf{x}], \mathbf{a}, \mathbf{b}), \max([\mathbf{x}], \mathbf{a}, \mathbf{b})], \\
&\qquad = \Big[[\mathbf{x}] \cup \mathbf{a} \cup \mathbf{b}\Big] \cap [\mathbf{c} \cup \mathbf{d}].\ \textit{(Eq. A.20)}
\end{aligned}
$$

Algorithm 15: $C_{\cap=\emptyset}([\mathbf{x}], \mathbf{a}, \mathbf{c}, \mathbf{d})$

Input: $[\mathbf{x}] = ([x_1], [x_2]), \mathbf{a} = (a_1, a_2), \mathbf{c} = (c_1, c_2), \mathbf{d} = (d_1, d_2)$

```
// Contraction over each component
```
$[x_1]^* = C_{\cap=\emptyset}([x_1], a_1, c_1, d_1)$;
$[x_2]^* = C_{\cap=\emptyset}([x_2], a_2, c_2, d_2)$;

Output: $[\mathbf{x}]^* = ([x_1]^*, [x_2]^*)$.

A.3 Segments and Convex Polygons

A.3.1 Definitions

This section present polygons as considered in this report.

A convex Polygon P corresponds to a convex subset of $\mathbb{R}^2$, delimited by at least three segments. We note n_P the number of edges (at least three) of the polygon P. The edges of P are denoted $\mathbf{p}_k$, $k = 1, \cdots, n_P$. Those edges are named according to a **trigonometric** order in this report.

Remark A.3.1 *A polygon P with n_P edges, is delimited by n_P segments. In order to facilitate the reading, the first edge $\mathbf{p}_1$ is equivalent to the edge $\mathbf{p}_{n_P+1}$. That leads to*

$$\bigcup_{k=1}^{n_P} Seg(\mathbf{p}_k, \mathbf{p}_{k+1}),\ \textit{with}\ \mathbf{p}_{n_P+1} \equiv \mathbf{p}_1. \tag{A.21}$$

Figure A.4 presents a convex polygon example.

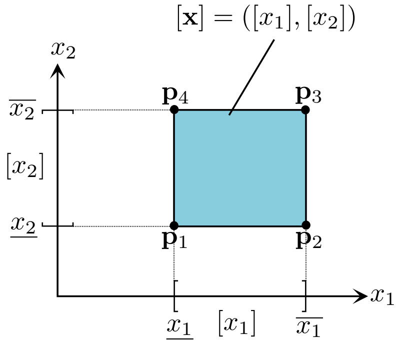


Figure A.5: A box is a convex polygon.

Definition A.3.1 *Let P be a polygon with n_P edges $\mathbf{p}_k$ $(k = 1, \cdots, n_P)$, then*

$$\forall \mathbf{x} \in P, \forall \mathbf{p}_k \in P, \det(\mathbf{x} - \mathbf{p}_k | \mathbf{p}_{k+1} - \mathbf{p}_k) \leq 0. \tag{A.22}$$

In other words, Equation A.22 describes all the points inside inside the polygon (border included). It can be noticed that this equation is correct only if the edges of the polygon are ordered in a trigonometric order. It is then possible to defined a polygon P as

$$P = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \bigwedge_{k=1}^{n_P} \det(\mathbf{x}_i - \mathbf{p}_k | \mathbf{p}_{k+1} - \mathbf{p}_k) \leq 0\}. \tag{A.23}$$

Remark A.3.2 *A two dimensional interval vector (a box) can be assimilated to a convex polygon. Let $[\mathbf{x}] = ([x_1], [x_2]) = ([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}])$ be a box:*

$$\begin{aligned}
[\mathbf{x}] = \{\mathbf{x}_i \in \mathbb{R}^2 \mid\ & \det(\mathbf{x}_i - (\underline{x_1}, \underline{x_2}) | (\overline{x_1}, \underline{x_2}) - (\underline{x_1}, \underline{x_2})) \leq 0 \ \wedge \\
& \det(\mathbf{x}_i - (\overline{x_1}, \underline{x_2}) | (\overline{x_1}, \overline{x_2}) - (\overline{x_1}, \underline{x_2})) \leq 0 \ \wedge \\
& \det(\mathbf{x}_i - (\overline{x_1}, \overline{x_2}) | (\underline{x_1}, \overline{x_2}) - (\overline{x_1}, \overline{x_2})) \leq 0 \ \wedge \\
& \det(\mathbf{x}_i - (\underline{x_1}, \overline{x_2}) | (\underline{x_1}, \underline{x_2}) - (\underline{x_1}, \overline{x_2})) \leq 0\}.
\end{aligned} \tag{A.24}$$

Figure A.5 illustrates this remark. Algorithm 16 allows to match a two dimensional box $\mathbb{R}^2$ with a convex polygon.

Algorithm 16: *Boite2Polygone*$([\mathbf{x}])$

Input: $[\mathbf{x}] = ([x_1], [x_2])$

$\mathbf{p}_1 = (\underline{x_1}, \underline{x_2})$;

$\mathbf{p}_2 = (\overline{x_1}, \underline{x_2})$;

$\mathbf{p}_3 = (\overline{x_1}, \overline{x_2})$;

$\mathbf{p}_4 = (\underline{x_1}, \overline{x_2})$;

$P_{\mathbf{x}}$ = polygon defined by the four edges $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{p}_4$;

Output: $P_{\mathbf{x}}$.

A.3.2 Polygon/Segment Intersection

Remark A.3.3 *Let P be a polygon with n_P edges, and $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^2$ two distinct points with $\mathbf{a} \notin P$ and $\mathbf{b} \notin P$. Then*

$$Seg(\mathbf{a}, \mathbf{b}) \cap P \neq \emptyset \Leftrightarrow Seg(\mathbf{a}, \mathbf{b}) \cap \Big(\bigcup_{k=1}^{n_P} Seg(\mathbf{p}_k, \mathbf{p}_{k+1}) \Big) \neq \emptyset. \tag{A.25}$$

Figure A.6 illustrates this remark.

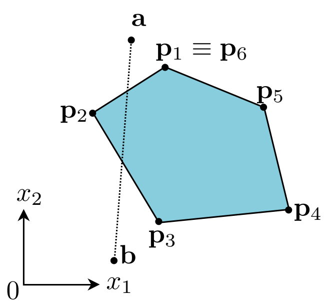


Figure A.6: Remark A.3.3 notifies that the segment $Seg(\mathbf{a}, \mathbf{b})$ intersects the polygon only if it intersects one of its segment border (knowing that **a** and **b** do not belong to the polygon). In this example $Seg(\mathbf{a}, \mathbf{b}) \cap Seg(\mathbf{p}_1, \mathbf{p}_2) \neq \emptyset$ and $Seg(\mathbf{a}, \mathbf{b}) \cap Seg(\mathbf{p}_2, \mathbf{p}_3) \neq \emptyset$.

Appendix B

Determinant Properties

B.1 First Proposition

Proposition B.1.1 *Let* $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ *and* $\mathbf{c} = (c_1, c_2)$ *be three distinct points of* $\mathbb{R}^2$. *Then*

$$\det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}) = \det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a}) = \det(\mathbf{b}-\mathbf{c}|\mathbf{a}-\mathbf{c}), \tag{B.1}$$
$$\det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}) = \det(\mathbf{c}-\mathbf{b}|\mathbf{a}-\mathbf{b}) = \det(\mathbf{b}-\mathbf{a}|\mathbf{c}-\mathbf{a}), \tag{B.2}$$
$$\det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}) = -\det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}). \tag{B.3}$$

Proof

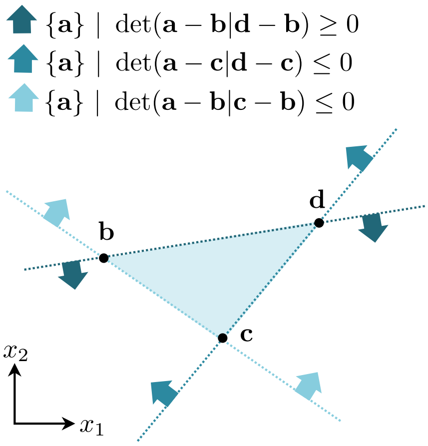
$$\begin{aligned}
\det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}) &= (a_1-b_1)(c_2-b_2)-(a_2-b_2)(c_1-b_1) \text{ (Eq. A.2)},\\
&= (a_1c_2-a_1b_2-b_1c_2+b_1b_2)-(a_2c_1-a_2b_1-b_2c_1+b_2b_1),\\
\det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}) &= a_1c_2-a_1b_2-b_1c_2-a_2c_1+a_2b_1+b_2c_1.\\
\det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a}) &= (c_1-a_1)(b_2-a_2)-(c_2-a_2)(b_1-a_1) \text{ (Eq. A.2)},\\
&= (c_1b_2-c_1a_2-a_1b_2+a_1a_2)-(c_2b_1-c_2a_1-a_2b_1+a_1a_2),\\
\det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a}) &= c_1b_2-c_1a_2-a_1b_2-c_2b_1+c_2a_1+a_2b_1,\\
\det(\mathbf{c}-\mathbf{a}|\mathbf{b}-\mathbf{a}) &= \det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}).
\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{b}-\mathbf{c}|\mathbf{a}-\mathbf{c}) &= (b_1-c_1)(a_2-c_2)-(b_2-c_2)(a_1-c_1) \text{ (Eq. A.2)},\\
&= (b_1a_2-b_1c_2-c_1a_2+c_1c_2)-(b_2a_1-b_2c_1-c_2a_1+c_2c_1),\\
\det(\mathbf{b}-\mathbf{c}|\mathbf{a}-\mathbf{c}) &= b_1a_2-b_1c_2-c_1a_2-b_2a_1+b_2c_1+c_2a_1,\\
\det(\mathbf{b}-\mathbf{c}|\mathbf{a}-\mathbf{c}) &= \det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}).
\end{aligned}$$

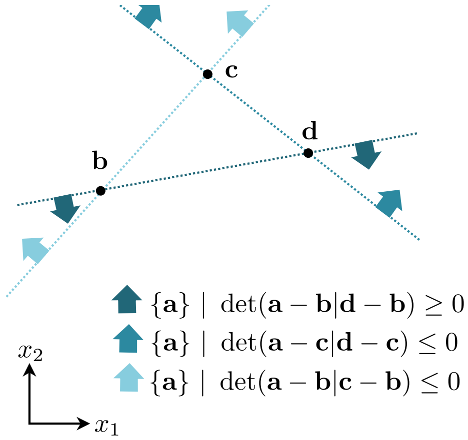
$$\begin{aligned}
\det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}) &= (a_1-c_1)(b_2-c_2)-(a_2-c_2)(b_1-c_1) \text{ (Eq. A.2)},\\
&= (a_1b_2-a_1c_2-c_1b_2+c_1c_2)-(a_2b_1-a_2c_1-c_2b_1+c_2c_1),\\
\det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}) &= a_1b_2-a_1c_2-c_1b_2-a_2b_1+a_2c_1+c_2b_1,\\
\det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}) &= -\det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}).
\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{c}-\mathbf{b}|\mathbf{a}-\mathbf{b}) &= (c_1-b_1)(a_2-b_2)-(c_1-b_2)(a_1-b_1) \text{ (Eq. A.2)},\\
&= (c_1a_2-b_1a_2-c_1b_2+b_1b_2)-(c_2a_1-c_2b_1-b_2a_1+b_2b_1),\\
\det(\mathbf{c}-\mathbf{b}|\mathbf{a}-\mathbf{b}) &= c_1a_2-b_1a_2-c_1b_2-c_2a_1+c_2b_1+b_2a_1,\\
\det(\mathbf{c}-\mathbf{b}|\mathbf{a}-\mathbf{b}) &= \det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}).
\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{b}-\mathbf{a}|\mathbf{c}-\mathbf{a}) &= (b_1-a_1)(c_2-a_2)-(b_2-a_2)(c_1-a_1) \text{ (Eq. A.2)},\\
&= (b_1c_2-b_1a_2-a_1c2+a_1a_2)-(b_2c_1-b_2a_1-a_2c_1+a_2a_1),\\
\det(\mathbf{b}-\mathbf{a}|\mathbf{c}-\mathbf{a}) &= b_1c_2-b_1a_2-a_1c_2-b_2c_1+b_2a_1+a_2c_1,\\
\det(\mathbf{b}-\mathbf{a}|\mathbf{c}-\mathbf{a}) &= \det(\mathbf{a}-\mathbf{c}|\mathbf{b}-\mathbf{c}).
\end{aligned}$$



(a) The inside of the triangle **bcd** corresponds to the points **a** that satisfy the three constraints. It can be noticed that $\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq 0$.



(b) In this case $\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) > 0$, but is does not exist any point **a** that satisfies the three constraints.

Figure B.1: Illustration of Proposition B.2.1.

B.2 Second Proposition

Proposition B.2.1 *Let* $\mathbf{a} = (a_1, a_2)$*,* $\mathbf{b} = (b_1, b_2)$*,* $\mathbf{c} = (c_1, c_2)$ *and* $\mathbf{d} = (d_1, d_2)$ *be four distinct points of* $\mathbb{R}^2$*. Then*

$$\begin{aligned} &\det(\mathbf{a}-\mathbf{b}|\mathbf{d}-\mathbf{b}) \geq 0 \ \wedge \ \det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}) \leq 0 \ \wedge \ \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq 0 \\ &\quad \Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq 0. \end{aligned} \tag{B.4}$$

Figure B.1 illustrates this proposition.

Proof

$$\begin{aligned}
&\begin{cases} \det(\mathbf{a}-\mathbf{b}|\mathbf{d}-\mathbf{b}) \geq 0 \\ \det(\mathbf{a}-\mathbf{b}|\mathbf{c}-\mathbf{b}) \leq 0 \\ \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq 0 \end{cases}, \\
&\Leftrightarrow \begin{cases} (a_1-b_1)(d_2-b_2)-(a_2-b_2)(d_1-b_1) \geq 0 \text{ (Eq. A.2)} \\ (a_1-b_1)(c_2-b_2)-(a_2-b_2)(c_1-b_1) \leq 0 \text{ (Eq. A.2)} \\ (a_1-c_1)(a_2-c_2)-(a_2-c_2)(d_1-c_1) \leq 0 \text{ (Eq. A.2)} \end{cases}, \\
&\Leftrightarrow \begin{cases} a_1d_2 - a_1b_2 - b_1d_2 - a_2d_1 + a_2b_1 + b_2d_1 \geq 0 \\ a_1c_2 - a_1b_2 - b_1c_2 - a_2c_1 + a_2b_1 + b_2c_1 \leq 0 \\ a_1d_2 - a_1c_2 - c_1d_2 - a_2d_1 + a_2c_1 + c_2d_1 \leq 0 \end{cases}, \\
&\Leftrightarrow \begin{cases} a_1d_2 - b_1d_2 - a_2d_1 + b_2d_1 \geq a_1b_2 - a_2b_1 \\ a_1c_2 - b_1c_2 - a_2c_1 + b_2c_1 \leq a_1b_2 - a_2b_1 \\ a_1d_2 - a_1c_2 - c_1d_2 - a_2d_1 + a_2c_1 + c_2d_1 \leq 0 \end{cases}, \\
&\Leftrightarrow \begin{cases} a_1d_2 - b_1d_2 - a_2d_1 + b_2d_1 \geq a_1b_2 - a_2b_1 \geq a_1c_2 - b_1c_2 - a_2c_1 + b_2c_1 \\ a_1d_2 - a_1c_2 - c_1d_2 - a_2d_1 + a_2c_1 + c_2d_1 \leq 0 \end{cases}, \\
&\Leftrightarrow \begin{cases} a_1d_2 - a_2d_1 - a_1c_2 + a_2c_1 \geq b_1d_2 - b_2d_1 - b_1c_2 + b_2c_1 \\ a_1d_2 - a_1c_2 - a_2d_1 + a_2c_1 \leq c_1d_2 - c_2d_1 \end{cases}, \\
&\Leftrightarrow c_1d_2 - c_2d_1 \geq a_1d_2 - a_2d_1 - a_1c_2 + a_2c_1 \geq b_1d_2 - b_2d_1 - b_1c_2 + b_2c_1, \\
&\Leftrightarrow c_1d_2 - c_2d_1 \geq b_1d_2 - b_2d_1 - b_1c_2 + b_2c_1, \\
&\Leftrightarrow b_1d_2 - b_1c_2 - c_1d_2 - b_2d_1 + b_2c_1 + c_2d_1 \leq 0, \\
&\Leftrightarrow (b_1-c_1)(d_1-c_2)-(b_2-c_2)(d_1-c_1) \leq 0, \\
&\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq 0 \text{ (Eq. A.2)}.
\end{aligned}$$

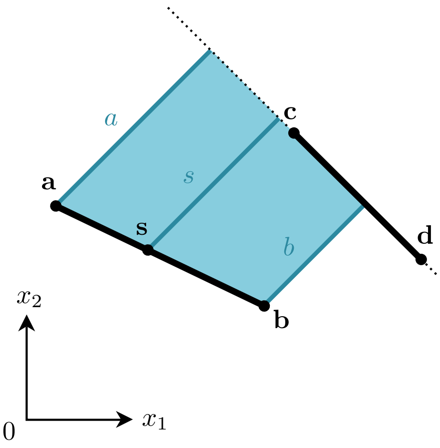


Figure B.2: Illustration of Proposition B.3.1: $a \geq b \Rightarrow a \geq s \geq b$

B.3 Third Proposition

Proposition B.3.1 *Let $Seg(\mathbf{a},\mathbf{b})$ be a segment with $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, $\mathbf{s} \in Seg(\mathbf{a},\mathbf{b})$ be a point, and $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ and $\mathbf{d} = (d_1, d_2) \in \mathbb{R}^2$ be two points with $Seg(\mathbf{a},\mathbf{b}) \cap Seg(\mathbf{c},\mathbf{d}) = \emptyset$. Then*

$$\begin{aligned} &\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \\ &\quad \Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}), \qquad \text{(B.5)} \\ &\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \\ &\quad \Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}). \qquad \text{(B.6)} \end{aligned}$$

In other words, Proposition B.3.1 notifies that the distance between the point $\mathbf{s}$ and the line associated to the vector $\overrightarrow{\mathbf{cd}}$ is bounded by the distances of the two edges of the segment $\mathbf{ab}$ and this line. This is illustrated Figure B.2.

Proof

$$\begin{aligned} \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= (a_1-c_1)(d_2-c_2) - (a_2-c_2)(d_1-c_1) \text{ (Eq. A.2)}, \\ \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= a_1d_2 - a_1c_2 - c_1d_2 - a_2d_1 + a_2c_1 + c_2d_1, \qquad \text{(B.7)} \\ \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= (b_1-c_1)(d_2-c_2) - (b_2-c_2)(d_1-c_1) \text{ (Eq. A.2)}, \\ \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= b_1d_2 - b_1c_2 - c_1d_2 - b_2d_1 + b_2c_1 + c_2d_1. \qquad \text{(B.8)} \end{aligned}$$

$$\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) = b_1d_2 - b_1c2 - b_2d_1 + b_2c_1 - a_1d_2 + a_1c_2 + a_2d_1 - a_2c_1. \qquad \text{(B.9)}$$

$$\begin{aligned} \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= (s_1-c_1)(d_2-c_2) - (s_2-c_2)(d_1-c_1) \text{ (Eq. A.2)}, \\ &= s_1d_2 - s_1c_2 - c_1d_2 - s_2d_1 + s_2c_1 + c_2d_1, \\ &= s_1(d_2-c_2) + s_2(c_1-d_1) + c_2d_1 - c_1d_2, \\ &= \Big((1-t)a_1 + tb_1\Big)(d_2-c_2) + \Big((1-t)a_2 + tb_2\Big)(c_1-d_1) \\ &\quad + c_2d_1 - c_1d_2 \text{ (Eq. A.1)}, \\ &= a_1d_2 - a_1c_2 - ta_1d_2 + ta_1c_2 + tb_1d_2 - tb_1c_2 + a_2c_1 - a_2d_1, \\ &\quad - ta_2c_1 + ta_2d_1 + tb_2c_1 - tb_2d_1 + c_2d_1 - c_1d_2, \\ &= t(a_1c_2 - a_1d_2 + b_1d_2 - b_1c_2 - a_2c_1 + a_2d_1 + b_2c_1 - b_2d_1), \\ &\quad + a_1d_2 - a_1c_2 + a_2c_1 - a_2d_1 - c_1d_2 + c_2d_1, \\ \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) &= t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big) \\ &\quad + \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \text{ (Eq. B.9 et B.7)}. \qquad \text{(B.10)} \end{aligned}$$

First case:

$$\begin{aligned}
&\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \geq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big)\ (t \geq 0,\ \text{Eq. A.1}),\\
&\quad\Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big) + \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.10}). \qquad \text{(B.11)}
\end{aligned}$$

$$\begin{aligned}
&\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big)\\
&\qquad (t \in [0,1],\ \text{Eq. A.1}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big) + \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.10}). \qquad \text{(B.12)}
\end{aligned}$$

Then, if $\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})$,

$$\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.11 et B.12}).$$

Second case:

$$\begin{aligned}
&\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \leq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big)\ (t \geq 0,\ \text{Eq. A.1}),\\
&\quad\Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big) + \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.10}). \qquad \text{(B.13)}
\end{aligned}$$

$$\begin{aligned}
&\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow 0 \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big)\\
&\qquad (t \in [0,1],\ \text{Eq. A.1}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq t\Big(\det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) - \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c})\Big) + \det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}),\\
&\quad\Leftrightarrow \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \geq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.10}). \qquad \text{(B.14)}
\end{aligned}$$

Then, if $\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})$,

$$\det(\mathbf{a}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{s}-\mathbf{c}|\mathbf{d}-\mathbf{c}) \leq \det(\mathbf{b}-\mathbf{c}|\mathbf{d}-\mathbf{c})\ (\text{Eq. B.13 et B.14}).$$

Appendix C

Some Proofs

In order to make the reading of this report easier, some proofs are written here.

C.1 Proof of Proposition 2.1.1

Proposition C.1.1 *Let* $\mathbf{x} \in \mathbb{R}^2$ *be a point and* $Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ *a segment with* $\mathbf{x} \notin Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$*. Then*

$$\begin{aligned} &\det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \cdot \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\det(\mathbf{e}_{1_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) \cdot \det(\mathbf{e}_{2_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) > 0 \\ \Leftrightarrow &\zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \vee \\ &\zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0. \end{aligned} \tag{C.1}$$

with

$$\zeta_x = \begin{cases} 1 & \textit{if } \det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \textit{otherwise.} \end{cases} \tag{C.2}$$

Proof We aim to prove that:

$$\begin{aligned} &\det(\mathbf{x} - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \cdot \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\det(\mathbf{e}_{1_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) \cdot \det(\mathbf{e}_{2_j} - \mathbf{x} | \mathbf{x}_i - \mathbf{x}) > 0 \\ \Leftrightarrow &\zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) > 0 \vee \\ &\zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) < 0. \end{aligned} \tag{C.3}$$

To make this easier to read, in the following, $\det(\mathbf{a} - \mathbf{b} | \mathbf{c} - \mathbf{b})$ is denoted $\mathrm{d}(\mathbf{a}, \mathbf{b}, \mathbf{c})$.
Two cases are possible: $\zeta_x = -1$ and $\zeta_x = 1$ (Eq. C.2).
<u>First case:</u>

$$\zeta_x = -1 \Leftrightarrow \mathrm{d}(\mathbf{x}, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \text{ (Eq: C.2).} \tag{C.4}$$

Then we want to prove:

$$\begin{aligned} &\mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{e}_{1_j}, \mathbf{x}, \mathbf{x}_i) \cdot \mathrm{d}(\mathbf{e}_{2_j}, \mathbf{x}, \mathbf{x}_i) > 0 \Leftrightarrow \\ &\mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) < 0 \ \vee \ \det(\mathbf{x}_i | \mathbf{e}_{2_j}) \mathbf{x} > 0 \text{ (Eq. C.3 et C.4).} \end{aligned}$$

$$\begin{aligned} &\mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \\ &\quad \Leftrightarrow d(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ d(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ d(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) > 0 \\ &\qquad \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) < 0, \\ &\quad \Leftrightarrow \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) > 0 \\ &\qquad \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) < 0 \ \wedge \ \mathrm{d}(\mathbf{x}, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \text{ (Eq. C.4),} \\ &\quad \Leftrightarrow \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) > 0 \\ &\qquad \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \text{ (Eq. B.4),} \\ &\mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \\ &\quad \Leftrightarrow \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{e}_{2_j}) < 0 \ \vee \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{2_j}, \mathbf{x}) > 0 \ \wedge \ \mathrm{d}(\mathbf{x}_i, \mathbf{e}_{1_j}, \mathbf{x}) > 0. \end{aligned} \tag{C.5}$$

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0,\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x},\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \text{ (Eq. C.4)},\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \text{ (Eq. B.4)},\\
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0.
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0,\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \text{ (Eq. C.5 and C.6)},\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})\cdot\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0.
\end{aligned}
$$

The Proposition C.1.1 is then verified when $\zeta_x=-1$.
<u>Second case:</u>

$$
\zeta_x=1 \Leftrightarrow \mathrm{d}(\mathbf{x},\mathbf{e}_{1_j},\mathbf{e}_{2_j})>0 \Leftrightarrow \mathrm{d}(\mathbf{x},\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \text{ (Eq: C.2 and B.3)}. \tag{C.7}
$$

We now aim to prove that:

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})>0 \;\vee\; \mathrm{d}(\mathbf{e}_{1_j},\mathbf{x},\mathbf{x}_i)\cdot\mathrm{d}(\mathbf{e}_{2_j},\mathbf{x},\mathbf{x}_i)>0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})>0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \text{ (Eq. C.3 and C.7)}\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \text{ (Eq. B.3)}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0,\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x},\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \text{ (Eq. C.7)},\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \text{ (Eq. B.4)},\\
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0.
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0,\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x},\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \text{ (Eq. C.7)},\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0\\
&\qquad\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \text{ (Eq. B.4)},\\
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{e}_{2_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0.
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
&\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0,\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})>0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0 \;\vee\\
&\qquad \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})<0 \;\wedge\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})<0 \text{ (Eq. C.8 and C.9)},\\
&\quad\Leftrightarrow \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{e}_{1_j})<0 \;\vee\; \mathrm{d}(\mathbf{x}_i,\mathbf{e}_{1_j},\mathbf{x})\cdot\mathrm{d}(\mathbf{x}_i,\mathbf{e}_{2_j},\mathbf{x})>0.
\end{aligned}
$$

Proposition C.1.1 is then verified when $\zeta_x = 1$.

C.2 Proof of Proposition 2.1.2

Proposition C.2.1 *Let $Seg(\mathbf{x}_1, \mathbf{x}_2)$ be a segment with $\mathbf{x}_1 \in \mathbb{R}^2$ and $\mathbf{x}_2 \in \mathbb{R}^2$, and $Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j})$ be a segment with $Seg(\mathbf{x}_1, \mathbf{x}_2) \cap Seg(\mathbf{e}_{1_j}, \mathbf{e}_{2_j}) = \emptyset$. Then*

$$
\begin{aligned}
&\forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0 \\
&\Leftrightarrow \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \geq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \geq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0.
\end{aligned}
$$

Proof According to Proposition B.3.1

$$
\begin{aligned}
&\zeta_x \in [\zeta_{x_1}, \zeta_{x_2}], \\
&\det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \in [\det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}), \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j})], \\
&\det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \in [\det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}), \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j})].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0 \\
&\Leftrightarrow [\zeta_{x_1}, \zeta_{x_2}] \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad [\zeta_{x_1}, \zeta_{x_2}] \cdot [\det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}), \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j})] \leq 0 \ \wedge \\
&\qquad [\zeta_{x_1}, \zeta_{x_2}] \cdot [\det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}), \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j})] \geq 0 \\
&\Leftrightarrow \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \ \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \\
&\qquad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0.
\end{aligned}
\tag{C.10}
$$

It can be noticed that

$$
\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \Leftrightarrow \zeta_{x_1} = \zeta_{x_2}.
$$

Then it can be deduced

$$
\begin{aligned}
&\zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \ \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \\
&\qquad \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \\
&\Leftrightarrow \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) \geq 0 \ \wedge \ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) \geq 0 \\
&\Leftrightarrow \forall \mathbf{x} \in Seg(\mathbf{x}_1, \mathbf{x}_2), \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{e}_{2_j} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \\
&\qquad \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x} - \mathbf{e}_{1_j}) \leq 0 \ \wedge \ \zeta_x \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x} - \mathbf{e}_{2_j}) \geq 0 \text{ (Eq. C.10).}
\end{aligned}
$$

C.3 Proof of Proposition 2.2.2

$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), (\mathbf{x}_i \mathrm{V} \mathbf{x}_s)_{\varepsilon_j^s}\} \\ &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\quad \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_s - \mathbf{e}_{1_j}) > 0 \ \vee\ \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_s - \mathbf{e}_{2_j}) < 0 \vee \\ &\quad [\mathbf{x}_i \cup \mathbf{x}_s] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\} \end{aligned} \tag{C.11}$$

with,

$$\zeta_s = \begin{cases} 1 & \text{if } \det(\mathbf{x}_s - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0, \\ -1 & \text{otherwise.} \end{cases} \tag{C.12}$$

It can be noticed that

$$\forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), [\mathbf{x}_i \cup \mathbf{x}_s] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \Leftrightarrow [\mathbf{x}_i \cup \mathbf{x}_1 \cup \mathbf{x}_2] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset \tag{C.13}$$

Then there are two ways of writing Equation C.11

$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)) &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), (\mathbf{x}_i \mathrm{V} \mathbf{x}_s)_{\varepsilon_j^s}\} \\ &= \{\mathbf{x}_i \in \mathbb{R}^2 \mid \forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ &\quad \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_s - \mathbf{e}_{1_j}) > 0 \ \vee\ \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_s - \mathbf{e}_{2_j}) < 0\} \cup \\ &\quad \{\mathbf{x}_i \in \mathbb{R}^2 | \mid [\mathbf{x}_i \cup \mathbf{x}_s] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\}. \end{aligned} \tag{C.14}$$

There are two cases: $\zeta_{x_1} = \zeta_{x_2}$ and $\zeta_{x_1} = -\zeta_{x_2}$.

C.3.1 First case: $\zeta_{x_1} = \zeta_{x_2}$

In this case

$$\begin{aligned}&\zeta_{x_1} = \zeta_{x_2} \Leftrightarrow \zeta_s = \zeta_{x_1} = \zeta_{x_2} \\ &\Rightarrow \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) = \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) = \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}),\end{aligned} \tag{C.15}$$

and

$$\det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_s - \mathbf{e}_{1_j}) \in [\det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_1 - \mathbf{e}_{1_j}) \cup \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_2 - \mathbf{e}_{1_j})], \tag{C.16}$$

$$\det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_s - \mathbf{e}_{2_j}) \in [\det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_1 - \mathbf{e}_{2_j}) \cup \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_2 - \mathbf{e}_{2_j})]. \tag{C.17}$$

Then

$$\begin{aligned}&\forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \Leftrightarrow \\ &\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \wedge\ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_2 - \mathbf{e}_{1_j}) > 0,\end{aligned} \tag{C.18}$$

$$\begin{aligned}&\forall \mathbf{x}_s \in Seg(\mathbf{x}_1,\mathbf{x}_2), \zeta_s \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \Leftrightarrow \\ &\quad \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \ \wedge\ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_2 - \mathbf{e}_{2_j}) > 0.\end{aligned} \tag{C.19}$$

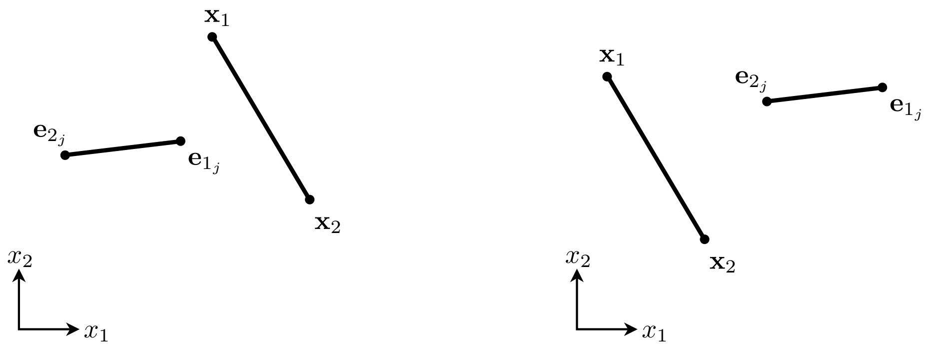
According to Equations C.18, C.19 et C.14,

$$\begin{aligned}\mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1,\mathbf{x}_2)) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid\ & \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{e}_{2_j} - \mathbf{e}_{1_j}) > 0 \vee \\ & \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \wedge\ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{1_j}|\mathbf{x}_2 - \mathbf{e}_{1_j}) > 0 \vee \\ & \zeta_{x_1} \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \ \wedge\ \zeta_{x_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j}|\mathbf{x}_2 - \mathbf{e}_{2_j}) > 0 \vee \\ & [\mathbf{x}_i \cup \mathbf{x}_s] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\}. \end{aligned} \tag{C.20}$$

C.3.2 Second Case: $\zeta_{x_1} = -\zeta_{x_2}$

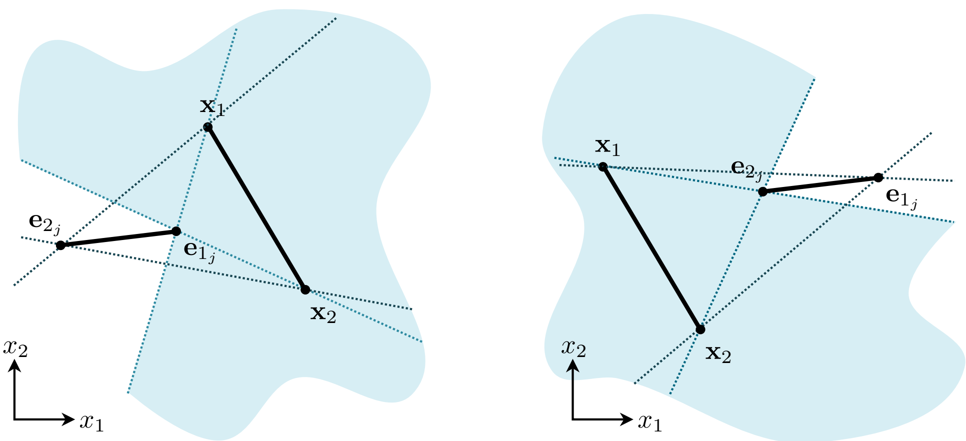
It has to be reminded that $Seg(\mathbf{x}_1,\mathbf{x}_2) \cap Seg(\mathbf{e}_{1_j},\mathbf{e}_{2_j}) = \emptyset$. Then it can be deduced that

$$\zeta_{x_1} = -\zeta_{x_2} \Rightarrow \zeta_{e_1} = \zeta_{e_2}. \tag{C.21}$$



(a) $\zeta_{e_1} = \zeta_{e_2} = 1$. (b) $\zeta_{e_1} = \zeta_{e_2} = -1$.

Figure C.1: The two possible configurations.



(a) $\zeta_{e_1} = \zeta_{e_2} = 1$. (b) $\zeta_{e_1} = \zeta_{e_2} = -1$.

Figure C.2: Visible spaces in both configurations.

It appears that there are two possible configurations: $\zeta_{e_1} = \zeta_{e_2} = 1$ (Figure C.1a) and $\zeta_{e_1} = \zeta_{e_2} = -1$ (Figure C.1b).

In the first configuration ($\zeta_{e_1} = \zeta_{e_2} = 1$) the visible space can be defined as (Figure C.2a):

$$\begin{aligned} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \\ & \big(\det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \vee \ \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) < 0\big) \wedge \\ & \big(\det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \ \vee \ \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) < 0\big) \vee \\ & [\mathbf{x}_i \cup \mathbf{x}_1 \cup \mathbf{x}_2] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\}. \end{aligned} \tag{C.22}$$

For the second configuration ($\zeta_{e_1} = \zeta_{e_2} = -1$) the visible space corresponds to (Figure C.2b):

$$\begin{aligned} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \\ & \big(\det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) < 0 \ \vee \ \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) > 0\big) \wedge \\ & \big(\det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) < 0 \ \vee \ \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) > 0\big) \vee \\ & [\mathbf{x}_i \cup \mathbf{x}_1 \cup \mathbf{x}_2] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\}. \end{aligned} \tag{C.23}$$

From that, it can be deduced that when $\zeta_{x_1} = -\zeta_{x_2}$,

$$\begin{aligned} \mathrm{E}_{\varepsilon_j^s}(Seg(\mathbf{x}_1, \mathbf{x}_2)) = \{\mathbf{x}_i \in \mathbb{R}^2 \mid \\ & \big(\zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_1 - \mathbf{e}_{1_j}) > 0 \ \vee \ \zeta_{e_1} \det(\mathbf{x}_i - \mathbf{e}_{1_j} | \mathbf{x}_2 - \mathbf{e}_{1_j}) < 0\big) \wedge \\ & \big(\zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_1 - \mathbf{e}_{2_j}) > 0 \ \vee \ \zeta_{e_2} \det(\mathbf{x}_i - \mathbf{e}_{2_j} | \mathbf{x}_2 - \mathbf{e}_{2_j}) < 0\big) \vee \\ & [\mathbf{x}_i \cup \mathbf{x}_1 \cup \mathbf{x}_2] \cap [\mathbf{e}_{1_j} \cup \mathbf{e}_{2_j}] = \emptyset\}. \end{aligned} \tag{C.24}$$