Algebraic properties of Manin matrices 1

A. Chervov, G. Falqui, Vladimir Roubtsov

Advances in Applied Mathematics 43 (2009) 239–315

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Algebraic properties of Manin matrices 1

A. Chervov [a], G. Falqui [b], V. Rubtsov [c,*]

[a] *Institute for Theoretical and Experimental Physics, Moscow, Russia*
[b] *Università di Milano – Bicocca, Milano, Italy*
[c] *Université D'Angers, Angers, France*

ARTICLE INFO

Article history:
Received 26 December 2008
Accepted 12 February 2009
Available online 16 June 2009

MSC:
primary 15A15
secondary 17B37, 15A23, 15A24, 15A33, 05A19, 05A30, 05E15, 13A50, 17B35, 20G05, 20G42, 16W35, 81R50

Keywords:
Determinant
Noncommutative determinant
Quasidetermininat
Permanent
Capelli identity
Turnbull identity
Representation theory
Quantum group
Right-quantum matrix
Cartier–Foata matrix
Manin matrix
Jacobi ratio theorem
MacMahon master theorem
Newton identities
Cayley–Hamilton theorem
Cramer's rule
Schur complement
Dodgson's condensation algorithm
Plucker coordinates

ABSTRACT

We study a class of matrices with noncommutative entries, which were first considered by Yu.I. Manin in 1988 in relation with quantum group theory. They are defined as "noncommutative endomorphisms" of a polynomial algebra. More explicitly their defining conditions read: (1) elements in the same column commute; (2) commutators of the cross terms are equal: $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$ (e.g. $[M_{11}, M_{22}] = [M_{21}, M_{12}]$). The basic claim is that despite noncommutativity many theorems of linear algebra hold true for Manin matrices in a form identical to that of the commutative case. Moreover in some examples the converse is also true, that is, Manin matrices are the most general class of matrices such that linear algebra holds true for them. The present paper gives a complete list and detailed proofs of algebraic properties of Manin matrices known up to the moment; many of them are new. In particular we provide complete proofs that an inverse to a Manin matrix is again a Manin matrix and for the Schur formula for the determinant of a block matrix; we generalize the noncommutative Cauchy–Binet formulas discovered recently arXiv:0809.3516, which includes the classical Capelli and related identities. We also discuss many other properties, such as the Cramer formula for the inverse matrix, the Cayley–Hamilton theorem, Newton and MacMahon–Wronski identities, Plücker relations, Sylvester's theorem, the Lagrange–Desnanot–Lewis Carroll formula, the Weinstein–Aronszajn formula, some multiplicativity properties for the determinant, relations with quasideterminants, calculation of the determinant via Gauss decomposition, conjugation to the second normal (Frobenius) form, and so on and so forth. Finally several examples and open question are discussed. We refer to [A. Chervov, G. Falqui, Manin matrices and Talalaev's formula, J. Phys. A 41 (2008) 194006; V. Rubtsov, A. Silantiev, D. Talalaev, Manin matrices, elliptic commuting families and characteristic polynomial of quantum gl_n elliptic Gaudin

0196-8858/$ – see front matter
doi:10.1016/j.aam.2009.02.003

model, in press] for some applications in the realm of quantum integrable systems.

Contents

* Corresponding author.
E-mail addresses: chervov@itep.ru (A. Chervov), gregorio.falqui@unimib.it (G. Falqui), volodya@tonton.univ-angers.fr (V. Rubtsov).

1. Introduction

It is well known that matrices with generically noncommutative elements do not admit a natural construction of the determinant with values in a ground ring and basic theorems of the linear algebra fail to hold true. On the other hand, matrices with noncommutative entries play a basic role in the theory of quantum integrability (see, e.g. [31]), in Manin's theory of "noncommutative symmetries" [76], and so on and so forth. Further we prove that many *results of commutative linear algebra* can be applied with minor modifications in the case of "Manin matrices".

We will consider the simplest case of those considered by Manin, namely – in the present paper – we will restrict ourselves to the case of commutators, and not of (super)-q-commutators, etc. Let us mention that Manin matrices are defined, roughly speaking, by *half* of the relations of the corresponding quantum group $Fun_q(GL(n))$ and taking $q=1$ (see Section 3.3).

Definition 1. Let M be an $n \times n'$ matrix with elements M_{ij} in (not necessarily commutative) ring $\mathcal{R}$. We will call M a Manin matrix if the following two conditions hold:

1. Elements in the same column commute between themselves.
2. Commutators of cross terms of 2×2 submatrices of M are equal:

$$[M_{ij}, M_{kl}] = [M_{kj}, M_{il}], \quad \forall i, j, k, l, \text{ e. g. } [M_{11}, M_{22}] = [M_{21}, M_{12}]. \tag{1.1}$$

A more intrinsic definition of Manin matrices via coaction on polynomial and Grassmann algebras will be recalled in Proposition 1 below. (Roughly speaking variables $\tilde{x}_i = \sum_j M_{ij} x_j$ commute among themselves if and only if M is a Manin matrix, where x_j are commuting variables, also commuting with elements of M).

In the previous paper [16] two of the authors have shown that Manin matrices have various applications in quantum integrability and outlined some of their basic properties. This paper is devoted solely to algebraic properties providing a complete amount of facts known up to the moment. Quite probably the properties established here can be transferred to some other classes of matrices with noncommutative entries, for example, to super-q-Manin matrices (see [20]) and quantum Lax matrices of most of the integrable systems. Such questions seems to be quite important for quantum integrability, quantum and Lie groups, as well as in the geometric Langlands correspondence theory [16,18]. But before studying these complicated issues, it seems to be worth to understand the simplest case in depth, this is one of the main motivations for us. The other one is that many statements are so simple and natural extension of the classical results, that can be of some interest just out of curiosity or pedagogical reasons for wide range of mathematicians.

1.1. Results and organization of the paper

The main aim of the paper is to argue the following claim: *linear algebra statements hold true for Manin matrices in a form identical to the commutative case.* Let us give a list of main properties discussed below. Some of these results are new, some – can be found in a previous literature: Yu. Manin has defined the determinant and has proven a Cramer's inversion rule, Laplace formulas, as well as Plücker identities; in S. Garoufalidis, T. Le, D. Zeilberger [39] the MacMahon–Wronski formula was proved; M. Konvalinka [66,67] found the Sylvester's identity and the Jacobi ratio's theorem, along with partial results on an inverse matrix and block matrices.[1] Some other results were announced in [16], where applications to integrable systems, quantum and Lie groups can be found.

- Section 3.4: Determinant can be defined in the standard way and it satisfies standard properties, e.g. it is completely antisymmetric function of columns and rows:

$$\det M = \det^{\mathrm{col}} M = \sum_{\sigma\in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{\sigma(i),i}. \tag{1.2}$$

- Proposition 4: let M be a Manin matrix and N satisfies: $\forall i, j, k, l$: $[M_{ij}, N_{kl}] = 0$:

$$\det^{\mathrm{col}}(MN) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(N). \tag{1.3}$$

  Let N be additionally a Manin matrix, then MN and $M \pm N$ are Manin matrices.
  Moreover in case $[M_{ij}, N_{kl}] \neq 0$, but obeys certain conditions we prove (Theorem 6):

$$\det^{\mathrm{col}}\big(MY + Q\ \mathrm{diag}(n-1, n-2, \dots, 1, 0)\big) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y). \tag{1.4}$$

  This generalizes [12] and the classical Capelli identity [11], here Q is a matrix related to the commutators of M_{ij}, Y_{kl} and $\mathrm{diag}(a_1, a_2, \dots, a_n)$ is a diagonal matrix with a_i on the diagonal.
- Section 4.1: Cramer's rule:

$$M^{-1} \text{ is a Manin matrix} \quad \text{and} \quad M^{-1}_{ij} = (-1)^{i+j}\det^{\mathrm{col}}(M)^{-1}\det^{\mathrm{col}}(\widehat{M_{ij}}). \tag{1.5}$$

  Here as usual $\widehat{M}_{lk}$ is the $(n-1)\times(n-1)$ submatrix of M obtained removing the lth row and the kth column.
- Section 7.1: the Cayley–Hamilton theorem: $\det^{\mathrm{col}}(t-M)|_{t=M} = 0$.
- Section 5.2: the formula for the determinant of block matrices:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}\big(D - CA^{-1}B\big) = \det^{\mathrm{col}}(D)\det^{\mathrm{col}}\big(A - BD^{-1}C\big). \tag{1.6}$$

  Also, we show that $D - CA^{-1}B$, $A - BD^{-1}C$ are Manin matrices. This is equivalent to the so-called Jacobi ratio theorem, stating that any minor of M^{-1} equals, up to a sign, to the product of $(\det^{\mathrm{col}} M)^{-1}$ and the corresponding complementary minor of the transpose of M.
- Section 7.2: the Newton and MacMahon–Wronski identities between $Tr\,M^k$, coefficients of $\det^{\mathrm{col}}(1-tM)$ and $Tr(S^kM)$. Denote by $\sigma(t), S(t), T(t)$ the following generating functions:

---
[1] These authors actually considered more general classes of matrices.

$$\sigma(t) = \det^{\mathrm{col}}(1 - tM), \qquad S(t) = \sum_{k=0,\dots,\infty} t^k \, Tr \, S^k M, \qquad T(t) = Tr \, \frac{M}{1 - tM}, \tag{1.7}$$

$$\text{then:} \quad 1 = \sigma(t)S(t), \qquad -\partial_t \sigma(t) = \sigma(t)T(t), \qquad \partial_t S(t) = T(t)S(t). \tag{1.8}$$

- Other facts are also discussed: Plücker relations (Section 7.3), Sylvester's theorem (Section 5.4), Lagrange–Desnanot–Lewis Carroll formula (Section 4.2), Weinstein–Aronszajn formula (Section 5.3), calculation of the determinant via Gauss decomposition (Section 7.4), conjugation to the second normal (Frobenius) form (Section 7.1), some multiplicativity properties for the determinant (Proposition 8), etc.
- Section 8.2: The matrix (Leningrad) notations form of the definition:

$$M \text{ is a Manin matrix} \tag{1.9}$$

$$\Leftrightarrow \quad [M \otimes 1, 1 \otimes M] = P[M \otimes 1, 1 \otimes M] \tag{1.10}$$

$$\Leftrightarrow \quad \frac{(1-P)}{2}(M \otimes 1)(1 \otimes M)\frac{(1-P)}{2} = \frac{(1-P)}{2}(M \otimes 1)(1 \otimes M). \tag{1.11}$$

- No-go facts: M^k is not a Manin matrix; elements $Tr(M)$, $\det^{\mathrm{col}}(M)$, etc. are not central. Moreover $[Tr(M), \det^{\mathrm{col}}(M)] \neq 0$ (Section 3.6.1); $\det^{\mathrm{col}}(e^M) \neq e^{Tr(M)}$, $\det^{\mathrm{col}}(1 + M) \neq e^{Tr(\ln(1+M))}$ (Section 12).

We also discuss relations with the quantum groups (Section 3.3) and mention some examples which are related to integrable systems, Lie algebras and quantum groups (Section 3.7).

Few more comments about the internal structure of the paper: Section 3 contains main definitions and properties. This material is crucial for what follows. The other sections can be read in an arbitrary order. We tried to make the exposition in each section independent of the others at least at formulations of the main theorems and notations. Though the proofs sometimes use the results from the previous sections. The short Section 2 is a kind if warm-up, which gives some simple examples to get the reader interested and to demonstrate some of results of the paper in the simplest possible case of 2×2 matrices. The content of each section can be read in the table of contents.

1.2. Context, history and related works

Manin matrices first appeared in [76], see also [21,75,77,78], where some basic facts like the determinant, Cramer's rule, Plücker identities etc. were established. The lectures [76] is the main source on the subject. Actually Manin's construction defines "noncommutative endomorphisms" of an arbitrary ring (and in principle of any algebraic structure). Here we restrict ourselves with the simplest case of the commutative polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, its "noncommutative endomorphisms" will be called Manin matrices. Some linear algebra facts were also established for a class of "good" rings (we hope to develop this in future). The main attention and application of the original works were on quantum groups, which are defined by "doubling" the set of the relations. The literature on "quantum matrices" (N. Reshetikhin, L. Takhtadzhyan, L. Faddeev) [98] is enormous – let us only mention [25,69] and especially D. Gurevich, A. Isaev, O. Ogievetsky, P. Pyatov, P. Saponov papers [53,97], where related linear algebra facts has been established for various quantum matrices.

Concerning "not-doubled" case let us mention the papers S. Wang [115], T. Banica, J. Bichon, B. Collins [4]. They investigate Manin's construction applied to finite-dimensional algebras for example to $\mathbb{C}^n$ (quantum permutation group). Such algebras appears to be C^*-algebras and are related to various questions in operator algebras. Linear algebra of such matrices is not known, (and, maybe it does not exist at all).

The simplest case which is considered here was somehow forgotten for many years after Manin's work (see however [99,100]). The situation changed after S. Garoufalidis, T. Le, D. Zeilberger [39], who discovered MacMahon–Wronski relations for q-Manin matrices. This result was followed by a flow of papers [28,36,54,68] etc.; let us in particular mention M. Konvalinka [66,67] which contains Sylvester's identity and Jacobi's ratio theorem, along with partial results on an inverse matrix and block matrices.

We came to this subject from another other direction. In [16] we observed that some examples of quantum Lax matrices in quantum integrability are exactly examples of Manin matrices. Moreover linear algebra of Manin matrices appears to have various applications in quantum integrability, quantum and Lie group theories. Numerous other quantum integrable systems provide various examples of matrices with noncommutative elements – quantum Lax matrices. This rises the question how to define the proper determinant and to develop linear algebra for such matrices. And, more generally does a proper notion of determinant exist? If yes, how to develop linear algebra? Such questions seem to be quite important for quantum integrability: they are related to the notion of "quantum spectral curve" which promises to be a key concept in the theory [17–20,102]. Manin's approach, applied for more general rings, provides classes of matrices where such questions can be possibly resolved. However we must remark that many examples from integrable systems do not fit in this approach.

From a more general point of view we deal with the question of the linear algebra of matrices with noncommutative entries. It should be remarked that the first appearance of a determinant for matrices with noncommutative entries goes back to A. Cayley. He was the first who had applied the notion of what we call *a column determinant* in the noncommutative setting. (We are thankful to V. Retakh for this remark). Let us mention the initial significative difference between our situation and the work I. Gelfand, S. Gelfand, V. Retakh, R. Wilson [45], where *generic* matrices with noncommutative entries are considered. There is no natural definition of the determinant (n^2 of "quasi-determinants" instead) in the "general noncommutative case" and their analogs of the linear algebra propositions are sometimes quite different from the commutative ones. Nevertheless their results of can be fruitfully applied to some questions here. Our approach is also different from the classical theory of J. Dieudonné [22] (see also [1]), since in this theory the determinant is an element of the quotient $\mathcal{K}^*/[\mathcal{K}^*,\mathcal{K}^*]$, where $\mathcal{K}^*$ is the multiplicative group of non-zero elements of the basic ring $\mathcal{K}$, while for Manin matrices the determinant is an element of the ring $\mathcal{K}$ itself.

We provide more detailed bibliographic notes in the text but we would like to add, as a *general disclaimer*, that our bibliographic notes are neither exhaustive nor historically ordered. We simply want to comment those papers and books that are more strongly related to our work.

We refer to "The Theory of Determinants in the Historical Order of Development" [86] for the early history of many results, which generalization to the noncommutative case are discussed below.

1.3. Remarks

In [39,66,67] the name "right quantum matrices" was used, in [71,99,100] the names "left" and "right quantum group", in [12] the name "row-pseudo-commutative". We prefer to use the name "Manin matrices".

All the considerations below work for an arbitrary field of characteristic not equal to 2, but we prefer to restrict ourselves with $\mathbb{C}$.

In subsequent papers [20] we plan to generalize the constructions below to the case of the Manin matrices related to the more general quadratic algebras as well as there applications to quantum integrable systems and some open problems.

2. Warm-up 2 × 2 examples: Manin matrices "everywhere"

Here we present some examples of the appearance of the Manin property in various very simple and natural questions concerning 2×2 matrices. The general idea is the following: we consider well-known facts of linear algebra and look how to relax the commutativity assumption for matrix elements such that the results will be still true. The answer is: *if and only if* M is a Manin matrix.

This section is a kind of warm-up, we hope to get the reader interested in the subsequent material and to demonstrate some results in the simplest examples. The expert reader may wish to skip this section.

Let us consider a 2×2 matrix M:

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

From Definition 1, M is a "Manin matrix" if the following commutation relations hold true:

- column-commutativity: $[a,c]=0$, $[b,d]=0$;
- equality of commutators of the cross-term: $[a,d]=[c,b]$.

The fact below can be considered as Manin's original idea about the subject.

Observation 1 *(Coaction on a plane).* Consider the polynomial ring $\mathbb{C}[x_1,x_2]$, and assume that the matrix elements a,b,c,d commute with x_1,x_2. Define $\tilde{x}_1,\tilde{x}_2$ by $\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$. Then $\tilde{x}_1,\tilde{x}_2$ commute among themselves *iff* M is a Manin matrix.

Proof.

$$[\tilde{x}_1,\tilde{x}_2]=[ax_1+bx_2,cx_1+dx_2]=[a,c]x_1^2+[b,d]x_2^2+\big([a,c]+[b,d]\big)x_1x_2. \qquad \square \tag{2.2}$$

Similar fact holds true for Grassmann variables (see Proposition 1 below).

Observation 2 *(Cramer rule).* The inverse matrix is given by the standard formula $M^{-1}=\frac{1}{ad-cb}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$ *iff* M is a Manin matrix.

Proof.

$$\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}da-bc & db-bd\\ -ca+ac & -cb+ad\end{pmatrix} \tag{2.3}$$

$$=\textit{iff } M \text{ is a Manin matrix}=\begin{pmatrix}ad-cb & 0\\ 0 & ad-cb\end{pmatrix}. \qquad \square \tag{2.4}$$

Observation 3 *(Cayley–Hamilton).* The equality $M^2-(a+d)M+(ad-cb)1_{2\times 2}=0$ holds *iff* M is a Manin matrix.

Proof.

$$\begin{aligned}
&M^2-(a+d)M+(ad-cb)1_{2\times 2}\\
&\quad=\begin{pmatrix}a^2+bc & ab+bd\\ ca+dc & cb+d^2\end{pmatrix}-\begin{pmatrix}a^2+da & ab+db\\ ac+dc & ad+d^2\end{pmatrix}+\begin{pmatrix}ad-cb & 0\\ 0 & ad-cb\end{pmatrix}\\
&\quad=\begin{pmatrix}(bc-da)+(ad-cb) & bd-db\\ ca-ac & 0\end{pmatrix}=\begin{pmatrix}[a,d]-[c,b] & [b,d]\\ [c,a] & 0\end{pmatrix}.
\end{aligned} \tag{2.5}$$

This vanishes *iff* M is a Manin matrix.

Let us mention that similar facts can be seen for the Newton identities, but not in such a strict form (see Example 12).

Observation 4 *(Multiplicativity of determinants (Binet Theorem)).* $\det^{\mathrm{col}}(MN)=\det^{\mathrm{col}}(M)\det(N)$ holds true for all $\mathbb{C}$-valued matrices N *iff* M is a Manin matrix.

$$\begin{aligned}
\det^{\mathrm{col}}(MN)-\det^{\mathrm{col}}(M)\det(N)&=[M_{11},M_{21}]N_{11}N_{12}+[M_{12},M_{22}]N_{21}N_{22}\\
&\quad+\big([M_{11},M_{22}]-[M_{21},M_{12}]\big)N_{21}N_{12}.
\end{aligned} \tag{2.6}$$

here and in the sequel, $\det^{\mathrm{col}}(X)$ is the column-determinant: $X_{11}X_{22} - X_{21}X_{12}$, i.e. elements from the first column stand first in each term.

2.1. Further properties in 2×2 case

Let us also present some other properties of 2×2 Manin matrices.

It is well known that in commutative case a matrix can be conjugated to the so-called Frobenius normal form. Let us show that the same is possible for Manin matrices, see also Section 7.1.

Observation 5 *(Frobenius form of a matrix).*

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -(ad-cb) & a+d \end{pmatrix}, \tag{2.7}$$

iff $[d,a] = [b,c]$ and $db = bd$.

Here it is enough to use two of three Manin's commutation relations. We shall see also the role the third relation in Example 11 of Section 7.

Let us denote the matrix in the right-hand side of (2.7) by M_{Frob} and the first matrix at the left-hand side by D.

To see that (2.7) is true we just write the following.

$$DM = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a^2+bc & ab+bd \end{pmatrix}. \tag{2.8}$$

$$M_{Frob}D = \begin{pmatrix} 0 & 1 \\ -(ad-cb) & a+d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ -(ad-cb)+a^2+da & ab+db \end{pmatrix} \tag{2.9}$$

$$= \begin{pmatrix} a & b \\ a^2+bc+([d,a]-[b,c]) & ab+bd+[d,b] \end{pmatrix}. \tag{2.10}$$

Observation 6 *(Inversion).* The two sided inverse of a Manin matrix M is Manin, and $\det^{\mathrm{col}}(M^{-1}) = (\det M)^{-1}$.

See Theorem 1 and Corollary 2.

Proof. Let us briefly prove this fact. From the Cramer's rule above, one knows the formula for the left inverse, which by assumption is also the right inverse. To prove the theorem one only needs to write explicitly that the right inverse is given by Cramer rule and the desired commutation relations appear automatically. Explicitly, from Cramer's formula (see Observation 2) we see that:

$$\frac{1}{ad-cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

One knows that if both left and right inverses exist then associativity guarantees that they coincide: $a_l^{-1} = a_l^{-1}(aa_r^{-1}) = (a_l^{-1}a)a_r^{-1} = a_r^{-1}$. So assuming that the right inverse to A exists, and denoting $ad - cb \equiv \delta$ we have:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\delta^{-1}) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a(\delta)^{-1}d - b(\delta)^{-1}c & -a(\delta)^{-1}b + b(\delta)^{-1}a \\ c(\delta)^{-1}d - d(\delta)^{-1}c & -c(\delta)^{-1}b + d(\delta)^{-1}a \end{pmatrix}. \tag{2.12}$$

Let us multiply the identity above by δ^{-1} on the left. We have:

$$(\delta)^{-1} = (\delta)^{-1}a(\delta)^{-1}d - (\delta)^{-1}b(\delta)^{-1}c, \quad \text{element (1,1)}, \tag{2.13}$$

$$(\delta)^{-1} = -(\delta)^{-1}c(\delta)^{-1}b + (\delta)^{-1}d(\delta)^{-1}a, \quad \text{element (2,2)}. \tag{2.14}$$

So we see that $(\det^{\mathrm{col}}(M))^{-1}$ equals to $\det^{\mathrm{col}}(M^{-1})$ and the last does not depend on the ordering of columns.

Moreover, equating (2.13) to (2.14) yields

$$(\delta)^{-1}a(\delta)^{-1}d - (\delta)^{-1}b(\delta)^{-1}c = -(\delta)^{-1}c(\delta)^{-1}b + (\delta)^{-1}d(\delta)^{-1}a, \tag{2.15}$$

$$\text{hence:} \quad \big[(\delta)^{-1}a, (\delta)^{-1}d\big] = \big[(\delta)^{-1}b, (\delta)^{-1}c\big]. \tag{2.16}$$

So the commutators of the cross-terms of M^{-1} are equal.

From the non-diagonal elements of equality 2.12 multiplied on the left by δ^{-1} we have:

$$(\delta)^{-1}c(\delta)^{-1}d - (\delta)^{-1}d(\delta)^{-1}c = 0, \qquad -(\delta)^{-1}a(\delta)^{-1}b + (\delta)^{-1}b(\delta)^{-1}a = 0,$$

$$\text{hence:} \quad \big[(\delta)^{-1}c, (\delta)^{-1}d\big] = 0, \qquad \big[(\delta)^{-1}a, (\delta)^{-1}b\big] = 0. \tag{2.17}$$

So we also have the column commutativity of the elements of M^{-1}. Hence the proposition is proved in 2×2 case. □

Observation 7 *(A puzzle with* $\det^{\mathrm{col}}(M) = 1$*).* Let M be a 2×2 Manin matrix, suppose that $\det^{\mathrm{col}}(M)$ is central element and it is invertible (for example $\det^{\mathrm{col}}(M) = 1$). Then all elements of M commute among themselves.

From the Observation 6 above one gets that

$$M^{-1} = \frac{1}{\det^{\mathrm{col}}(M)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{2.18}$$

is again a Manin matrix. This gives the commutativity of all elements. It is quite a surprising fact that imposing only one condition we "kill" the three commutators: $[a, b]$, $[a, d] = [c, b]$, $[c, d]$.

3. Manin matrices. Definitions and elementary properties

In this section we recall the definition of Manin matrices and describe their basic properties in the general $(n \times n)$ case. The material is rather a simple one, but it is necessary for the sequel. First we will give an explicit definition of Manin matricesin terms of commutation relations, and then we will provide a more conceptual point of view which defines them by the coaction property on the polynomial and Grassmann algebras. (This is the original point of view of Manin.) We also explain the relation of Manin matrices with quantum groups. As it was shown by Yu. Manin there exists a natural definition of the determinant which satisfies most of the properties of commutative determinants; this will be also recalled below. The main reference for this part is Yu. Manin's book [76], as well as [21,75,77,78].

3.1. Definition

Definition 2. Let us call a matrix M with elements in an associative ring $\mathcal{K}$ a "*Manin matrix*" if the properties below are satisfied:

- elements which belong to the same column of M commute among themselves;
- commutators of cross terms are equal: $\forall p,q,k,l$ $[M_{pq}, M_{kl}] = [M_{kq}, M_{pl}]$, e.g. $[M_{11}, M_{22}] = [M_{21}, M_{12}]$, $[M_{11}, M_{2k}] = [M_{21}, M_{1k}]$.

Remark 1. The second condition for the case $q = l$ obviously implies the first one. Nevertheless we deem it more convenient for the reader to formulate it explicitly.

The conditions can be restated as: for each 2×2 submatrix:

$$\begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & a & \ldots & b & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & c & \ldots & d & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad \text{of } M \text{ it holds } [a,d] = [c,b],\ [a,c] = 0 = [b,d]. \tag{3.1}$$

Obviously, by a "submatrix" we mean a matrix obtained as an intersection of two rows (i.e. straight horizontal lines, no decline) and two columns (i.e. straight vertical lines, no decline).

Remark 2. These relations were written by Yu. Manin [76] (see Chapter 6.1, Formula 1, page 37). Implicitly they are contained in Yu. Manin [75] – the last sentence on page 198 contains a definition of the algebra $end(A)$ for an arbitrary quadratic Koszul algebra A. One can show (see the remarks on the page 199 top) that $end(\mathbb{C}[x_1,\ldots,x_n])$ is the algebra generated by M_{ij}.

Remark 3. Actually, a matrix M such that its transposed M^t is a Manin matrix satisfies the same properties like Manin matrices do with some obvious modifications.

We will explicitly consider this class of M.

3.1.1. Poisson version of Manin matrices

Definition 3. An algebra $\mathcal{K}$ over $\mathbb{C}$ is called a Poisson algebra, if it is a commutative algebra, endowed with a bilinear antisymmetric operation, denoted as $\{*,*\}: R\otimes R \to R$ and called a Poisson bracket, such that the operation satisfies the Leibniz and the Jacobi identities (i.e. $\forall f,g,h \in R: \{fg,h\} = f\{g,h\} + \{f,h\}g$, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$).

Definition 4. We call "Poisson–Manin" a matrix M with elements in the Poisson algebra $\mathcal{K}$, such that $\{M_{ij}, M_{kl}\} = \{M_{kj}, M_{il}\}$.

We briefly discuss Poisson–Manin matrices in Section 8.3.

3.2. Characterization via coaction. Manin's construction

Here we recall Manin's original definition. It provides a conceptual approach to Manin matrices. Let us mention that the construction below is a specialization of Manin's general considerations. (See Yu. Manin [75–77].)

Proposition 1. *Coaction. Consider a rectangular $n \times m$-matrix M, the polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$ and the Grassmann algebra $\mathbb{C}[\psi_1, \dots, \psi_n]$ (i.e. $\psi_i^2 = 0$, $\psi_i \psi_j = -\psi_j \psi_i$); let x_i and ψ_i commute with M_{pq}: $\forall i, p, q$: $[x_i, M_{pq}] = 0$, $[\psi_i, M_{pq}] = 0$. Consider new variables $\tilde{x}_i$, $\tilde{\psi}_i$:*

$$\begin{pmatrix} \tilde{x}_1 \\ \dots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix}, \qquad (\tilde{\psi}_1, \dots, \tilde{\psi}_m) = (\psi_1, \dots, \psi_n) \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nm} \end{pmatrix}. \tag{3.2}$$

Then the following three conditions are equivalent:

- *M is a Manin matrix,*
- *the variables $\tilde{x}_i$ commute among themselves: $[\tilde{x}_i, \tilde{x}_j] = 0$,*
- *the variables $\tilde{\psi}_i$ anticommute among themselves: $\tilde{\psi}_i \tilde{\psi}_j + \tilde{\psi}_j \tilde{\psi}_i = 0$.*

Remark 4. The conditions $\tilde{\psi}_i^2 = 0$ are equivalent to column commutativity, and $\tilde{\psi}_i \tilde{\psi}_j = -\tilde{\psi}_j \tilde{\psi}_i$, $i < j$, to the cross term relations.

3.3. q-analogs and $RTT = TTR$ quantum group matrices

One can define q-analogs of Manin matrices and characterize their relation to quantum group theory. Actually q-Manin matrices are defined by half of the relations of the corresponding quantum group $Fun_q(GL_n)$[2] [98]. The remaining half consists of relations insuring that also M^t is a q-Manin matrix.

Definition 5. Let us call an $n \times n'$ matrix M by *q-Manin matrix*, if the following conditions hold true. For any 2×2 submatrix $(M_{ij,kl})$, consisting of rows i and k, and columns j and l (where $1 \leqslant i < k \leqslant n$, and $1 \leqslant j < l \leqslant n'$):

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & M_{ij} & \dots & M_{il} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & M_{kj} & \dots & M_{kl} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \equiv \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a & \dots & b & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & c & \dots & d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.3}$$

the following commutation relations hold:

$$ca = qac \quad (q\text{-commutation of the entries in a column}), \tag{3.4}$$

$$db = qbd \quad (q\text{-commutation of the entries in a column}), \tag{3.5}$$

$$ad - da = +q^{-1}cb - qbc \qquad (\text{cross commutation relation}). \tag{3.6}$$

In terms of M_{ij} this reads ($i < k$, $j < l$):

$$M_{kj} M_{ij} = q M_{ij} M_{kj}, \qquad M_{ij} M_{kl} - M_{kl} M_{ij} = q^{-1} M_{kj} M_{il} - q M_{il} M_{kj}. \tag{3.7}$$

For $q = 1$ this definition reduces to the Definition 2 of Manin matrices.

[2] More precisely we should write $Fun_q(Mat_n)$, since we do not localize the q-determinant.

Definition 6. An $n \times n$ matrix T belongs to the quantum group $Fun_q(GL_n)$ if the following conditions hold true. For any 2×2 submatrix $(T_{ij,kl})$, consisting of rows i and k, and columns j and l (where $1 \leqslant i < k \leqslant n$, and $1 \leqslant j < l \leqslant n$):

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & T_{ij} & \dots & T_{il} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & T_{kj} & \dots & T_{kl} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \equiv \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a & \dots & b & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & c & \dots & d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.8}$$

the following commutation relations hold:

$$ca = qac \quad (q\text{-commutation of the entries in a column}), \tag{3.9}$$

$$db = qbd \quad (q\text{-commutation of the entries in a column}), \tag{3.10}$$

$$ba = qab \quad (q\text{-commutation of the entries in a row}), \tag{3.11}$$

$$dc = qcd \quad (q\text{-commutation of the entries in a row}), \tag{3.12}$$

$$ad - da = +q^{-1}cb - qbc \quad (\text{cross commutation relation } 1), \tag{3.13}$$

$$bc = cb \quad (\text{cross commutation relation } 2). \tag{3.14}$$

As quantum groups are usually defined within the so-called matrix (Leningrad) formalism, let us briefly recall it. (We will further discuss this issue in Section 8).

Lemma 1. *The commutation relations for quantum group matrices can be described in matrix (Leningrad) notations as follows*:

$$R(T \otimes 1)(1 \otimes T) = (1 \otimes T)(T \otimes 1)R, \tag{3.15}$$

where R-matrix can be given, for example, by the formula:

$$R = q^{-1} \sum_{i=1,\dots,n} E_{ii} \otimes E_{ii} + \sum_{i,j=1,\dots,n;\ i \neq j} E_{ii} \otimes E_{jj} + \left(q^{-1} - q\right) \sum_{i,j=1,\dots,n;\ i>j} E_{ij} \otimes E_{ji}, \tag{3.16}$$

where E_{ij} are standard matrix units – zeroes everywhere except 1 in the intersection of the ith row with the jth column.

For example in 2×2 case the R-matrix is:

$$R = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{3.17}$$

Remark 5. This R-matrix differs by the change $q \to q^{-1}$ from the one in [98], formula 1.5, page 185.

The relation between q-Manin matrices and quantum groups consists in the following simple proposition:

Proposition 2. *A matrix T is a matrix in the quantum group $Fun_q(GL_n)$ if and only if T and simultaneously the transpose matrix T^t are q-Manin matrices.*

So one sees that Manin matrices can be seen as characterized by a "half" of the conditions that characterize the corresponding quantum matrix group.

q-Manin matrices can be characterized by the coaction on a q-polynomial and a q-Grassmann algebra in the same way as in $q = 1$ case. Here is the analogue of Proposition 1.

Proposition 3. *Consider a rectangular $n \times m$-matrix M, the q-polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$, where $\forall i < j$: $x_i x_j = q^{-1} x_j x_i$, i.e. $\forall i, j$: $x_i x_j = q^{sgn(i-j)} x_j x_i$ and the q-Grassmann algebra $\mathbb{C}[\psi_1, \dots, \psi_n]$ (i.e. $\psi_i^2 = 0$, $\psi_i \psi_j = -q \psi_j \psi_i$, for $i < j$; i.e. $\forall i, j$: $\psi_i \psi_j = -q^{-sgn(i-j)} \psi_j \psi_i$); suppose x_i and ψ_i commute with the matrix elements M_{pq}. Consider the variables $\tilde{x}_i$, $\tilde{\psi}_i$ defined by*:

$$\begin{pmatrix} \tilde{x}_1 \\ \dots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix}, \qquad (\tilde{\psi}_1, \dots, \tilde{\psi}_m) = (\psi_1, \dots, \psi_n) \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nm} \end{pmatrix}, \tag{3.18}$$

that is the new variables are obtained via left action (in the polynomial case) and right action (in the Grassmann case) of M on the old ones. Then the following three conditions are equivalent:

- *The matrix M is a q-Manin matrix.*
- *The variables $\tilde{x}_i$ q-commute among themselves: $\forall i < j$: $\tilde{x}_i \tilde{x}_j = q^{-1} \tilde{x}_j \tilde{x}_i$, i.e. $\forall i, j$: $\tilde{x}_i \tilde{x}_j = q^{sgn(i-j)} \tilde{x}_j \tilde{x}_i$.*
- *The variables $\tilde{\psi}_i$ q-anticommute among themselves*: $\tilde{\psi}_i^2 = 0$, $\tilde{\psi}_i \tilde{\psi}_j = -q \tilde{\psi}_j \tilde{\psi}_i$, *for $i < j$; i.e. $\forall i, j$: $\tilde{\psi}_i \tilde{\psi}_j = -q^{-sgn(i-j)} \tilde{\psi}_j \tilde{\psi}_i$.*

The conditions $\tilde{\psi}_i^2 = 0$ are equivalent to the relations (3.4), (3.5), and the conditions $\tilde{\psi}_i \tilde{\psi}_j = -q \tilde{\psi}_j \tilde{\psi}_i$, $i < j$, are equivalent to the relations (3.6).

We plan to discuss q-Manin matrices and some of their applications in the theory of integrability in a subsequent publications (see [20]).

3.4. The determinant

Here we recall a definition of the determinant following Manin's ideas. It is well known that, for generic matrices over a noncommutative ring (possibly, algebra), one cannot develop a full theory of determinants with values in the same ring. However, for some specific matrices there may exist a "good" notion of determinant. In particular for Manin matrices one can define the determinant just taking the column expansion as a definition. Despite its simplicity such a definition is actually a good one. It satisfies almost all the properties of the determinants in the commutative case and is consistent with the other concepts of noncommutative determinants (quasideterminants of I. Gelfand, V. Retakh [43] (see Section 4.1.1) and Dieudonné [22] determinant). Lemma 2 provides a more conceptual approach to the notion of determinant. It states that the determinant equals to a coefficient of the action on the top form, exactly in the same as in the commutative case.

Definition 7. Let M be a Manin matrix. Define the determinant of M by column expansion:

$$\det M = \det^{\mathrm{col}} M = \sum_{\sigma \in S_n} (-1)^\sigma \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{\sigma(i) i}, \tag{3.19}$$

where S_n is the group of permutations of n letters, and the symbol $\curvearrowright$ means that in the product $\prod_{i=1,\ldots,n} M_{\sigma(i),i}$ one writes at first the elements from the first column, then from the second column and so on and so forth.

Example 1. For the case $n=2$, we have

$$\det^{\mathrm{col}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\mathrm{def}}{=} ad - cb \stackrel{\text{Lemma 3}}{=} da - bc. \tag{3.20}$$

The second equality is a restatement of the second condition of Definition 2.

Let us now recall the setting of Section 3.2. Consider a Grassmann algebra $\mathbb{C}[\psi_1,\ldots,\psi_n]$ (i.e. $\psi_i^2=0$, $\psi_i\psi_j=-\psi_j\psi_i$); let ψ_i commute with M_{pq}: $\forall i,p,q$: $[\psi_i, M_{pq}]=0$. Consider the new variables $\tilde{\psi}_i$:

$$(\tilde{\psi}_1,\ldots,\tilde{\psi}_n) = (\psi_1,\ldots,\psi_n)\begin{pmatrix} M_{11} & \ldots & M_{1n} \\ \ldots & & \\ M_{n1} & \ldots & M_{nn} \end{pmatrix}. \tag{3.21}$$

By Proposition 1 we have that

$$\tilde{\psi}_i\tilde{\psi}_j = -\tilde{\psi}_j\tilde{\psi}_i. \tag{3.22}$$

Lemma 2. *For an arbitrary matrix M (not necessarily a Manin matrix) it holds*:

$$\det^{\mathrm{col}}(M)\psi_1\wedge\cdots\wedge\psi_n = \tilde{\psi}_1\wedge\cdots\wedge\tilde{\psi}_n. \tag{3.23}$$

If M is a Manin matrix, it is true that:

$$\forall p\in S_n\quad \det^{\mathrm{col}}(M)\psi_1\wedge\cdots\wedge\psi_n = (-1)^{sgn(p)}\tilde{\psi}_{p(1)}\wedge\cdots\wedge\tilde{\psi}_{p(n)}. \tag{3.24}$$

The proof is straightforward.

Lemma 3. *The exchange of any two columns in a Manin matrix changes only the sign of the determinant. More generally: an arbitrary permutation p of columns changes the determinant only by multiplication by $(-1)^{sgn(p)}$, that is, the determinant is a fully antisymmetric function of columns of a Manin matrix.*

This property is specific for Manin matrices. For generic matrices the column determinant can be defined, but it does not satisfy this basic property.

Proof of Lemma 3. Let us denote a Manin matrix by M and by M^p the matrix obtained by the permutation of columns with respect to the permutation $p\in S_n$. It is quite easy to see that for an arbitrary matrix M, (not necessarily a Manin matrix) it is true that:

$$\tilde{\psi}_{p(1)}\wedge\cdots\wedge\tilde{\psi}_{p(n)} = \det^{\mathrm{col}}\bigl(M^p\bigr)\psi_1\wedge\cdots\wedge\psi_n. \tag{3.25}$$

For Manin matrices due to anticommutativity of $\tilde{\psi}_i$ (which is guaranteed by Proposition 1) $\tilde{\psi}_{p(1)}\wedge\cdots\wedge\tilde{\psi}_{p(n)} = (-1)^{sgn(p)}\tilde{\psi}_1\wedge\cdots\wedge\tilde{\psi}_n$ which equals to $(-1)^p\det^{\mathrm{col}}(M)\tilde{\psi}_1\wedge\cdots\wedge\tilde{\psi}_n$ by Lemma 2. So we conclude that $\det^{\mathrm{col}}(M^p) = (-1)^{sgn(p)}\det^{\mathrm{col}}(M)$, for an arbitrary permutation $p\in S_n$. $\square$

Another proof of this fact goes as follows. Since any permutation can be presented as a product of transpositions of neighbours $(i, i+1)$ it is enough to prove the proposition for such transpositions. But for them it follows from the equality of the commutators of cross elements (formula (1.1)).

Lemma 4. *The exchange of two* rows *in an* arbitrary *matrix (not necessarily a Manin matrix) changes only the sign of* $\det^{\mathrm{col}}$. *Also if two rows in a matrix M coincide, then* $\det^{\mathrm{col}}(M) = 0$. *Also, one can add any row to another row and this does not change the determinant. More generally one can add a row multiplied by a scalar (or more generally multiplied by an element which commutes with M_{ij}) and the same is true.*

The proof of these statements is immediate. Let us stress that no conditions of commutativity is necessary.

The lemmas above imply the following:

Corollary 1.

1. *Assume that two columns or two rows in a Manin matrix M coincide, then* $\det^{\mathrm{col}}(M) = 0$.
2. *One can add any column of a Manin matrix to another column and this does not change the determinant. More generally one can add a column multiplied by a scalar (or more generally multiplied by an element which commutes with M_{ij}) and the same is true.*
3. *One can easily see that any submatrix of a Manin matrix is a Manin matrix. So one has natural definition of minors and again one can choose an arbitrary order of columns (rows) to define minors.*
4. *The determinant of a Manin matrix does not depend on the order of columns in the column expansion, i.e.*

$$\forall p \in S_n \quad \det^{\mathrm{col}} M = \sum_{\sigma \in S_n} (-1)^{\sigma} \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{\sigma(p(i))\ p(i)}. \tag{3.26}$$

Proposition 4. *Multiplicativity. Let A be a Manin matrix and A' a generic matrix with elements in the same ring $\mathcal{K}$. Suppose that all elements of A' commute with all elements of A, i.e. $\forall i, j, k, l$: $[A_{ij}, A'_{kl}] = 0$. Then* $\det^{\mathrm{col}}(AA') = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(A')$. *If A' is also a Manin matrix, then AA' is a Manin matrix.*

Remark 6. If A, B are the matrices such that $\forall i, j, k, l [A_{ij}, B_{kl}] = 0$, (for example B is a $\mathbb{C}$-valued matrix), it nevertheless does not follow that $\det^{\mathrm{col}}(AB) = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(B)$, even in 2×2 case.

Proof. The proposition is a direct consequence of the coaction characterization of Manin matrices (Proposition 1) The details are as follows.

Consider the Grassmann algebra $\Lambda[\psi_1, \dots, \psi_n]$ and introduce $\tilde{\psi}_1, \dots, \tilde{\psi}_n$, $\tilde{\tilde{\psi}}_1, \dots, \tilde{\tilde{\psi}}_n$ as

$$\begin{gathered}(\tilde{\psi}_1, \dots, \tilde{\psi}_n) = (\psi_1, \dots, \psi_n) \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \dots & & \\ A_{n1} & \dots & A_{nn} \end{pmatrix}, \\ (\tilde{\tilde{\psi}}_1, \dots, \tilde{\tilde{\psi}}_n) = (\tilde{\psi}_1, \dots, \tilde{\psi}_n) \begin{pmatrix} A'_{11} & \dots & A'_{1m} \\ \dots & & \\ A'_{n1} & \dots & A'_{nm} \end{pmatrix}. \end{gathered} \tag{3.27}$$

It is easy to see that (see Lemma 2):

$$\tilde{\psi}_1 \wedge \dots \wedge \tilde{\psi}_n = \det^{\mathrm{col}}(A) \psi_1 \wedge \dots \wedge \psi_n. \tag{3.28}$$

This equality does not require anything except anticommutativity of ψ_i and $[A_{ij}, \psi_l] = 0$. (In particular we do not need A to be a Manin matrix). However, when A is a Manin matrix, by Proposition 1, $\tilde{\psi}_i$

also anticommute; since we require that $\forall i, j, k, l$: $[A_{ij}, A'_{kl}] = 0$ and $[\psi_k, A'_{kl}] = 0$, we can use the same lemma again:

$$\tilde{\tilde{\psi}}_1 \wedge \cdots \wedge \tilde{\tilde{\psi}}_n = \det^{\mathrm{col}}(A')\tilde{\psi}_1 \wedge \cdots \wedge \tilde{\psi}_n. \tag{3.29}$$

Using 3.28:

$$\tilde{\tilde{\psi}}_1 \wedge \cdots \wedge \tilde{\tilde{\psi}}_n = \det^{\mathrm{col}}(A')\det^{\mathrm{col}}(A)\psi_1 \wedge \cdots \wedge \psi_n. \tag{3.30}$$

On the other hand one can apply Lemma 2 directly to the product $(A'A)$:

$$\tilde{\tilde{\psi}}_1 \wedge \cdots \wedge \tilde{\tilde{\psi}}_n = \det^{\mathrm{col}}(A'A)\psi_1 \wedge \cdots \wedge \psi_n. \tag{3.31}$$

Combining the equalities one gets $\det^{\mathrm{col}}(A'A) = \det^{\mathrm{col}}(A')\det^{\mathrm{col}}(A)$, which is equal to $\det^{\mathrm{col}}(A)\det^{\mathrm{col}}(A')$ since $\forall i, j, k, l$: $[A_{ij}, A'_{kl}] = 0$. So the first part of the proposition is proved.

Whenever A' is a Manin matrix as well, one sees that $\tilde{\tilde{\psi}}_i$ are again Grassmann variables (Proposition 1). On the other hand $\tilde{\tilde{\psi}}_i = \sum_l \psi_l (AA')_{li}$. So by the same proposition AA' is a Manin matrix. □

We can also argue $\det^{\mathrm{col}}(AA') = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(A')$ in more direct way. One should observe that *all* elements of $\det^{\mathrm{col}}(A)\det^{\mathrm{col}}(A')$ are contained in $\det^{\mathrm{col}}(AA')$, but generally written in different order – the property that $\det^{\mathrm{col}}(A)$ does not depend on the order of column expansion provides that one can reorder in an appropriate way. The property of the column commutativity of elements of A provides that unwanted terms in $\det^{\mathrm{col}}(AA')$ cancel each other.

We have already seen (Observation 4) that in 2×2 case $\det^{\mathrm{col}}(AA') = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(A')$ for any A' implies that A is a Manin matrix. The straightforward generalization is not true in $n \times n$ case. Indeed, consider a matrix A such that all elements in some row are equal to zero. Then, clearly $\det^{\mathrm{col}}(A) = 0 = \det^{\mathrm{col}}(AA')$ for any matrix A'. In 2×2 case any matrix with row of zeroes is a Manin matrix, however this is clearly not true for 3×3 matrices, etc. However for generic enough matrices A, such that $\det^{\mathrm{col}}(A) = \det^{\mathrm{col}}(AA')$ for any $\mathbb{C}$-valued A', it is true that A is a Manin matrix. This can be seen considering A' to be transposition matrices and matrices $1 + E_{ii+1}$, where as usually E_{ij} matrix unit with zeroes everywhere except 1 at position (ij). However we do not know how to formulate this "generality condition" in a compact form.

Remark 7. Since $\det^{\mathrm{col}} M^t = \det^{\mathrm{row}} M$, where $\det^{\mathrm{row}} M = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1,\ldots,n} M_{i\sigma(i)}$, the statements above can be easily reformulated for the case M^t is a Manin matrix.

3.5. The permanent

The permanent of a matrix is a polylinear function of its columns and rows, similar to the determinant, without sign factors $(-1)^{sign(\sigma)}$ in its definition. We will make use of it in Section 7.2.2 below.

Definition 8. Let M be a Manin matrix. We define its permanent by *row* expansion[3] as

$$\mathrm{perm} M = \mathrm{perm}^{\mathrm{row}} M = \sum_{\sigma \in S_n} \overset{\curvearrowright}{\prod_{i=1,\ldots,n}} M_{i\sigma(i)}, \tag{3.32}$$

[3] Remark the difference with the definition of the determinant, where one uses column expansion.

Lemma 5. *The permanent of a Manin matrix does not depend on the order of rows in the row expansion*:

$$\forall p \in S_n, \quad \mathrm{perm}^{\mathrm{row}} M = \sum_{\sigma \in S_n} \overset{\curvearrowright}{\prod_{i=1,\dots,n}} M_{p(i)\,\sigma(p(i))}, \tag{3.33}$$

or, in the other words, the permanent of a Manin matrix does not change after arbitrary permutation of rows in a matrix M:

$$\forall p \in S_n, \quad \mathrm{perm}^{\mathrm{row}} M = \mathrm{perm}^{\mathrm{row}} M^p, \tag{3.34}$$

where M^p *is a matrix obtained by the* p*-permutation of rows in matrix* M.

Proof. Since any permutation can be presented as a product of transpositions of neighbours $(i, i+1)$ it is enough to prove the proposition for such transpositions. But for them it follows from the equality of the commutators of cross elements (formula 1.1). □

Example 2.

$$\mathrm{perm}^{\mathrm{row}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\mathrm{def}}{=} ad + bc = da + cb, \tag{3.35}$$

where the last equation follows both form the lemma above and from the very definition of Manin matrix.

Remark 8. It is easy to see that the row-permanent of an arbitrary matrix M (even without conditions of commutativity) does not change under any permutation of columns.

3.6. Elementary properties

Let us herewith collect some properties described above, that are simple consequences of the definition of Manin matrix.

1. Any matrix with commuting elements is a Manin matrix.
2. Any submatrix of a Manin matrix is again a Manin matrix.
3. If A, B are Manin matrices and $\forall i, j, k, l$: $[A_{ij}, B_{kl}] = 0$, then $A + B$ is again a Manin matrix.
4. If A is a Manin matrix, c is a scalar, then cA is a Manin matrix.
5. If A is a Manin matrix, C is scalar matrix, then CA and AC are Manin matrices and $\det^{\mathrm{col}}(CA) = \det^{\mathrm{col}}(AC) = \det^{\mathrm{col}}(C)\det^{\mathrm{col}}(A)$.
6. If A, B are Manin matrices and $\forall i, j, k, l$: $[A_{ij}, B_{kl}] = 0$, then AB is a Manin matrix and $\det^{\mathrm{col}}(AB) = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(B)$. (Proposition 4).
7. If A is a Manin matrix, then one can exchange the ith and the jth columns (rows); one can put ith column (row) on jth place (erasing jth column (row)); one can add new column (row) to matrix A which is equal to one of the columns (rows) of the matrix A; one can add the ith column (row) multiplied by any constant to the jth column (row); in all cases the resulting matrix will be again a Manin matrix.
8. If A and simultaneously A^t are Manin matrices, then all elements A_{ij} commute with each other. (A q-analog of this lemma says that if A and simultaneously A^t are q-Manin, then A is quantum matrix: "$R\overset{1}{A}\overset{2}{A} = \overset{2}{A}\overset{1}{A}R$" (see Yu. Manin [75,76]).)
9. The exchange of two columns in a Manin matrix changes the sign of the determinant. If two columns or two rows in a Manin matrix M coincide, then $\det^{\mathrm{col}}(M) = 0$.

This has been already discussed in Corollary 1.

3.6.1. Some No-Go facts for Manin matrices

Let M be a Manin matrix with elements in the associative ring $\mathcal{K}$.

Fact. In general $\det^{col}(M)$ is not a central element of $\mathcal{K}$.

Remark 9. This should be compared with the quantum matrix group $Fun_q(GL_n)$, where $\det_q$ is central. The reason why this property does not hold for Manin matrices is that their defining relations are half of those of quantum matrix groups.

Fact. In general $[Tr\,M^k, Tr\,M^m] \neq 0$, $[Tr\,M, \det^{col}(M)] \neq 0$.

Remark 10. Taking traces of powers of Lax matrices is the standard procedure for obtaining commuting integrals of motion for integrable systems. Indeed, Manin's conditions (or their q-analogs) do not imply commutativity of traces. Although the concept of Manin matrix is related with (quantum) integrable systems, for this commutativity property one needs stronger conditions like the Yang–Baxter relation $R\overset{1}{T}\overset{2}{T} = \overset{2}{T}\overset{1}{T}R$.

Fact. In general $M^k, k = 2, \ldots,$ is not a Manin matrix. We will prove however that M^{-1} is again a Manin matrix.

Remark 11. For "quantum matrices" T such that [98] T: $R_q\overset{1}{T}\overset{2}{T} = \overset{2}{T}\overset{1}{T}R_q$, with R_q a solution of the Yang–Baxter equation it is true that T^k satisfies $R_{\tilde{q}}\overset{1}{T}\overset{2}{T} = \overset{2}{T}\overset{1}{T}R_{\tilde{q}}$, with $\tilde{q} = q^k$ (see, e.g. [77], Section 4.2.9, pages 132–133).

Fact. Let M be a Manin matrix; then in general $\det^{col}(e^M) \neq e^{Tr(M)}$, $\log(\det^{col}(M)) \neq Tr(\log(M))$ (see Section 12).

3.7. Examples

Definition 9. A matrix A with the elements in $\mathcal{K}$ is called a Cartier–Foata (see [13,35]) matrix if elements from different *rows* commute with each other.

Proposition 5. *Any Cartier–Foata matrix is a Manin matrix.*

Proof. Clear from the definitions. $\square$

Consider arbitrary elements $r_1, \ldots, r_n$ in a unital ring $\mathcal{K}$. The matrix

$$\begin{pmatrix} r_1 & r_2 & \ldots & r_n \\ r_1 & r_2 & \ldots & r_n \\ \ldots & \ldots & \ldots & \ldots \\ r_1 & r_2 & \ldots & r_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} \otimes \begin{pmatrix} r_1 & r_2 & \ldots & r_n \end{pmatrix} \tag{3.36}$$

is obviously a Manin matrix.

Let also M be an arbitrary Manin matrix, and consider elements c^i_{jk} such that they commute with each other and with all elements M (for example c^i_{jk} are scalars). Then

$$C^1 M C^2 + C^3, \tag{3.37}$$

where $(C^i)_{jk} = c^i_{jk}$, as well as its submatrices are Manin matrices.

Observation 8. Let M be a Manin matrix over $\mathcal{K}$, and consider an arbitrary invertible element $x \in R$. Then the matrix $(\widetilde{M})_{ij} = xM_{ij}x^{-1}$ is also obviously a Manin matrix.

Examples related to Lie algebras and integrable systems. Let us give some remarkable examples of Manin matrices. They are related to integrable systems and Lie algebras (see [16], Section 3 for further information).

Let x_{ij}, y_{ij} be commutative variables. Let X, Y be $n \times k$ matrices with matrix elements $(X)_{ij} = x_{ij}$, $(Y)_{ij} = y_{ij}$. Let us denote by ∂_X, ∂_Y the $n \times k$ matrices with matrix elements $\frac{\partial}{\partial x_{ij}}$ and $\frac{\partial}{\partial y_{ij}}$. Let z be a variable commuting with y_{ij}.

(i) The following $2n \times 2k$, $(n+k) \times (n+k)$ matrices are Manin matrices (the second one is related to the Capelli identities (see [16], Section 4.2.2)):

$$\begin{pmatrix} X & \partial_Y \\ Y & \partial_X \end{pmatrix}, \qquad \begin{pmatrix} z1_{k\times k} & (\partial_Y)^t \\ Y & \partial_z 1_{n\times n} \end{pmatrix}. \tag{3.38}$$

(ii) Let K^1, K^2 be $n \times n$, respectively $k \times k$ matrices with elements in $\mathbb{C}$, one can see that the following matrix is actually a Manin matrix:

$$\partial_z 1_{n\times n} + K^1 - Y\left(z1_{k\times k} + K^2\right)^{-1}(\partial_Y)^t. \tag{3.39}$$

For the sake of concreteness, we notice that, in the case $n = 2$, $k = 1$ the formula above yields the matrix:

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_z \end{pmatrix} + \begin{pmatrix} K^1_{11} & K^1_{12} \\ K^1_{21} & K^1_{22} \end{pmatrix} - \begin{pmatrix} \frac{y_1 \partial_{y_1}}{z-k} & \frac{y_1 \partial_{y_2}}{z-k} \\ \frac{y_2 \partial_{y_1}}{z-k} & \frac{y_2 \partial_{y_2}}{z-k} \end{pmatrix}. \tag{3.40}$$

(iii) Consider the standard matrix units e_{ij} (i.e. $n \times n$ matrices defined as $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$[4]). Consider a variable z and the operator ∂_z. These elements commute with e_{ij} and satisfy $[\partial_z, z] = 1$, $e^{-\partial_z} f(z) = f(z-1)e^{-\partial_z}$. Then

$$\partial_z 1_{n\times n} - \frac{1}{z}\begin{pmatrix} e_{11} & \dots & e_{1n} \\ \dots & \dots & \dots \\ e_{n1} & \dots & e_{nn} \end{pmatrix}, \qquad f(z)\, e^{-\partial_z}\left(1_{n\times n} + \frac{1}{z}\begin{pmatrix} e_{11} & \dots & e_{1n} \\ \dots & \dots & \dots \\ e_{n1} & \dots & e_{nn} \end{pmatrix}\right). \tag{3.41}$$

are Manin matrices, where $f(z)$ is an arbitrary function.

To check that the matrices in (3.38) are indeed Manin matrices, one only needs to use the standard commutation relations $[\frac{\partial}{\partial x_{ij}}, x_{kl}] = \delta_{ik}\delta_{jl}$, $[x_{ij}, x_{kl}] = [\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}}] = 0$ (the same for y_{ij} and z, while it is understood that operators referring to different "letters" commute), e.g. $[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{kl}}] = 0$. Similarly, for the matrix (3.40).

The matrices in (3.41) read, in the 2×2 and $f(z) = 1$ case:

$$\begin{pmatrix} \partial_z - \frac{1}{z}e_{11} & -\frac{1}{z}e_{12} \\ -\frac{1}{z}e_{21} & \partial_z - \frac{1}{z}e_{22} \end{pmatrix}, \qquad \begin{pmatrix} e^{-\partial_z}(1 + \frac{1}{z}e_{11}) & e^{-\partial_z}\frac{1}{z}e_{12} \\ e^{-\partial_z}\frac{1}{z}e_{21} & e^{-\partial_z}(1 + \frac{1}{z}e_{22}) \end{pmatrix}. \tag{3.42}$$

[4] In other words, e_{ij} has 1 in the position i, j and 0 everywhere else.

Let us check the Manin relations for the leftmost of these matrices: Column 1 commutativity reads[5]:

$$\left[\partial_z - \frac{1}{z}e_{11}, -\frac{1}{z}e_{21}\right] = -\left[\partial_z, \frac{1}{z}\right]e_{21} + \frac{1}{z^2}[e_{11}, e_{21}] = \frac{1}{z^2}e_{21} + \frac{1}{z^2}(-e_{21}) = 0. \tag{3.43}$$

The cross-term relation $[M_{11}, M_{22}] = [M_{21}, M_{12}]$ is:

$$[M_{11}, M_{22}] = \left[\partial_z - \frac{1}{z}e_{11}, \partial_z - \frac{1}{z}e_{22}\right] = -\left[\partial_z, \frac{1}{z}\right]e_{22} - e_{11}\left[\frac{1}{z}, \partial_z\right] = \frac{1}{z^2}e_{22} - \frac{1}{z^2}e_{11}, \tag{3.44}$$

$$[M_{21}, M_{12}] = \left[-\frac{1}{z}e_{21}, -\frac{1}{z}e_{12}\right] = \frac{1}{z^2}(e_{22} - e_{12}). \tag{3.45}$$

Notice that the matrices above are matrices with elements in $Diff(z) \otimes Mat_n$, i.e. they belong to $Mat_n[Diff(z) \otimes Mat_n]$, where $Diff(z)$ is an algebra of differential operators in z. Actually one only needs that elements e_{ij} satisfy the commutation relations $[e_{ij}, e_{kl}] = e_{il}\delta_{kj} - e_{kj}\delta_{il}$. For example, if one regards the e_{ij} as elements of the universal enveloping algebra of gl_n (or their images in an arbitrary representation), then the matrices (3.41) will still be Manin matrices. We should stress that in the examples above we consider $n \times n$ matrices with elements in $Mat_n \otimes Diff(z)$, and *not* $n^2 \times n^2$ matrices with elements in $Diff(z)$. Indeed, in the second case, they do not satisfy the Manin properties.

Remark 12. The examples above are intimately related to the Gaudin and XXX-Heisenberg integrable systems (see [16] and also [19]). According to a specialization of the remarkable result of D. Talalaev [110], the differential operators $H_i(z)$ defined as $\det^{\mathrm{col}}(M) = \sum_{k=0,\ldots,n} H_k(z)\partial_z^k$ (where M is defined by Eq. (3.39)) commute among themselves (i.e. $\forall i, j$: $[H_i(z), H_j(w)] = 0$). The operators $H_i(z)$ provide a full set of quantum commuting integrals of motion for the Gaudin integrable system. The gl_n Gaudin system was introduced in [40], but the full set of quantum integrals of motion (the so-called "higher Gaudin Hamiltonians"), whose existence was proved in [32] and whose commutativity proved in [26] was not explicitly constructed before [110]. This construction has far-reaching applications to Bethe ansatz and separation of variables (see [16,18]). Similar constructions also play an important role in the Langlands correspondence and Kac–Moody algebra theory (explicit description of the center of $U(\widehat{gl_n})$ [16–18]). Further considerations concerning q- and elliptic analogs will appear in [20,102].

3.8. Hopf structure

Let us consider the algebra over $\mathbb{C}$ generated by M_{ij} $1 \leqslant i, j \leqslant n$ with relations: $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$. One can see that it is a bialgebra with the coproduct $\Delta(M_{ij}) = \sum_k M_{ik} \otimes M_{kj}$. This is usually denoted as follows:

$$\Delta(M) = M \dot{\otimes} M. \tag{3.46}$$

Remark 13. It is easy to see that this coproduct is coassociative (i.e. $(\Delta \otimes 1) \otimes \Delta = (1 \otimes \Delta) \otimes \Delta$) and also (from Proposition 4), that it holds $\Delta(\det^{\mathrm{col}}(M)) = \det^{\mathrm{col}}(M) \otimes \det^{\mathrm{col}}(M)$.

The natural antipode for this bialgebra should be $S(M) = M^{-1}$. So it exists only in some "field of fractions" for the algebra generated by M_{ij}.

Remark 14. Let us frame the above property within the notions of noncommutative geometry (see, e.g. [76] for an introduction).

[5] Column 2 commutativity is completely analogous.

Consider a group G, and denote by $Fun(G)$ the algebras of functions on G. The multiplication map:

$$m : G \times G \to Fun(G), \tag{3.47}$$

clearly induces the dual map:

$$\Delta : Fun(G) \to Fun(G) \times Fun(G), \tag{3.48}$$

by the rule $\Delta(f)[g_1, g_2] = f(g_1 g_2)$. Associativity of the multiplication clearly induces coassociativity of the comultiplication: $(\Delta \otimes 1) \otimes \Delta = (1 \otimes \Delta) \otimes \Delta$.

So, according to the noncommutative geometry point of view, one should think of bialgebras as some kind of "functions" on "noncommutative spaces" which are not just spaces, but groups (more precisely semi-groups, since we have not discussed inversion operation, which corresponds to the antipode). So the algebra generated by matrix elements M_{ij} of a Manin matrix can be thought as the algebra of functions on some noncommutative (semi)group-space. Moreover this analogy can be continued. Consider a group G acting on some set V, and denote by $Fun(G), Fun(V)$ the algebras of functions on F and, respectively, V. The action of the group G on V defines a dual morphism of commutative algebras:

$$\phi : Fun(V) \to Fun(G) \otimes Fun(V), \qquad \phi(f)(g, v) = f(gv). \tag{3.49}$$

The condition $g_1(g_2 v) = (g_1 g_2)v$, implies:

$$(\Delta \otimes 1)(\phi) = (1 \otimes \phi)(\phi). \tag{3.50}$$

Which is now reformulated only in terms of ϕ, Δ, so makes sense for an arbitrary bialgebra.

The "coaction"-Proposition 1 implies that there exists morphisms of algebras:

$$\phi_1 : \mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}\langle M_{ij}\rangle \otimes \mathbb{C}[x_1, \ldots, x_m], \qquad \phi_1(x_i) = \sum_k M_{ik} x_k, \tag{3.51}$$

$$\phi_2 : \mathbb{C}[\psi_1, \ldots, \psi_n] \to \mathbb{C}\langle M_{ij}\rangle \otimes \mathbb{C}[\psi_1, \ldots, \psi_n], \qquad \phi_1(\psi_i) = \sum_k M_{ki} \psi_k. \tag{3.52}$$

One can check that both of the maps satisfy the condition: $(\Delta \otimes 1)(\phi_i) = (1 \otimes \phi_i)(\phi_i)$, $i = 1, 2$.

So one can consider the maps ϕ_i as "coactions" of Manin matrices on a the space $\mathbb{C}^n$ and its super version.

Remark 15. Let us also mention that there exists another coproduct. To motivate it let us give another look on the algebra generated by M_{ij}. The defining relations for M_{ij} are written entirely in terms of commutators, so the associative algebra generated by M_{ij} is the universal enveloping algebra to the Lie algebra defined by the relations $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$. For an arbitrary universal enveloping algebra the coproduct can be defined by the formula $\Delta(M_{ij}) = M_{ij} \otimes 1 + 1 \otimes M_{ij}$. It is easy to see directly (or conclude from the general properties of Lie algebras) that this coproduct is compatible with the defining relations, coassociative and there exist an antipode $S(M_{ij}) = -M_{ij}$ and counit $\epsilon(M_{ij}) = 0$, $\epsilon(1) = 1$. However this coproduct is not natural in Manin's framework (it is not compatible with the coaction on $\mathbb{C}[x_1, \ldots, x_n]$ described above).

4. Inverse of a Manin matrix

In this section we discuss several facts about the inverse of a Manin matrix. The first one is Cramer's rule which states that an inverse matrix can be calculated by the same formula with minors as in the commutative case; the second one is that inverse matrix is actually again a Manin matrix under natural conditions – this fact is related to a formula which goes back to Lagrange, Desnanot, Jacobi and Lewis Carroll in the commutative case. These should be considered among the main results of the paper.[6]

4.1. Cramer's formula and quasideterminants

Proposition 6. *(See [76].) Let M be a Manin matrix and denote by M^{adj} the adjoint matrix defined in the standard way, (i.e. $M^{adj}_{kl} = (-1)^{k+l}\det^{\mathrm{col}}(\widehat{M}_{lk})$) where $\widehat{M}_{lk}$ is the $(n-1)\times(n-1)$ submatrix of M obtained removing the lth row and the kth column. Then the same formula as in the commutative case holds true, that is*

$$M^{adj}M = \det^{\mathrm{col}}(M)1_{n\times n}. \tag{4.1}$$

Here $1_{n\times n}$ is the identity matrix of size n. If M^t is a Manin matrix, then M^{adj} is defined by row-determinants and $MM^{adj} = \det^{\mathrm{col}}(M^t)1_{n\times n} = \det^{\mathrm{row}}(M)1_{n\times n}$.

Proof. One can see that the equality $(M^{adj}M)_{ii} = \det^{\mathrm{col}}(M)$ $\forall i$ follows from the fact that $\det^{\mathrm{col}}(M)$ does not depend on the order of the column expansion in the determinant. This independence was proved above (Corollary 1). Let us introduce a matrix $\widetilde{M}$ as follows. Take the matrix M and set the ith column equal to the jth column; denote the resulting matrix by $\widetilde{M}$. Note that $\det^{\mathrm{col}}(\widetilde{M}) = 0$ precisely gives $(M^{adj}M)_{ij} = 0$ for $i \neq j$. To prove that $\det^{\mathrm{col}}(\widetilde{M}) = 0$ we argue as follows. Clearly $\widetilde{M}$ is a Manin matrix. Lemma 3 allows to calculate the determinant taking the elements first from ith column, then jth, then other elements from the other columns. This yields that $\det^{\mathrm{col}}(\widetilde{M}) = 0$, since it is the sum of the elements of the form $(xy - yx)(z) = 0$, where x, y are the elements from the ith and jth of $\widetilde{M}$, so from jth column of M. By column commutativity of a Manin matrix, $xy - yx = 0$, so $\det^{\mathrm{col}}(\widetilde{M}) = 0$. □

Remark 16. The only difference with the commutative case is that, in the equality (4.1) the order of the products of M^{adj} and M has to be kept in mind.

Remark 17. In the works by Manin (see, e.g. [76]) one can find wider classes of matrices with noncommutative entries with properly defined determinants and versions of the Cramer rule.

The question how far and whether the property $M^{adj}M = \det^{\mathrm{col}}(M)1_{n\times n}$ characterizes Manin matrices is open. Observation 2 shows that it is indeed the case for 2×2 matrices. Since in higher rank case the relations coming from the Cramer rule are of order n, while Manin's relations are always quadratic, it is not obvious at all how to settle the matter.

4.1.1. Relation with quasideterminants

We will herewith recall a few constructions from the theory of quasideterminants and discuss their counterparts in the case of Manin matrices. It is fair to say that the general theoretical set-up of quasideterminants I. Gelfand, S. Gelfand, V. Retakh, R. Wilson [41,43,45], can be briefly presented as follows: *many facts of linear algebra can be reformulated with the only use of an inverse matrix*. Thus it can be extended to the noncommutative setup and can be applied, for example, to some questions considered here. We must stress the difference between our set-up and that of [45]: we consider *a special class* of matrices with noncommutative entries (the Manin matrices), and for this class we can

[6] These results, together with sketchy proofs, were announced in [16].

extend many facts of linear algebra basically in the same form as in the commutative case, (in particular, as we have seen, there exists a well-defined notion of determinant). On the other hand, in [45] *generic matrices* are considered; thus there is no natural notion of the determinant, and facts of linear algebra are not exactly given in the same form as in the commutative case.

Let us recall [45, Definition 1.2.2, page 9], that the (p,q)th quasideterminant $|A|_{pq}$ of an invertible matrix A is defined as $|A|_{pq} = (A^{-1}_{qp})^{-1}$, i.e. the inverse to the (q,p)-element of the matrix inverse to A. It is also denoted by

$$|A|_{pq} = \begin{vmatrix} A_{11} & A_{12} & \dots & \dots & \dots & A_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \boxed{A_{pq}} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}. \tag{4.2}$$

From the Cramer rule we have

Lemma 6.

$$|M|_{pq} = (-1)^{p+q} \det^{\mathrm{col}}(\widehat{M}_{pq})^{-1} \det^{\mathrm{col}}(M), \tag{4.3}$$

$\widehat{M}_{pq}$ is the $(n-1)\times(n-1)$ submatrix of M obtained by removing the pth row and the qth column.

Also, from Lemma 12 below, one can deduce the

Lemma 7.

$$|A|_{pq} = A_{pq} - A_{p*}(\widehat{A}_{pq})^{-1} A_{*q}, \tag{4.4}$$

where $\widehat{A}_{pq}$ is the $(n-1)\times(n-1)$ submatrix of A obtained removing the pth row and the qth column, A_{p*} is pth row of A without the element A_{pq} and A_{*q} is qth column of A without the element A_{pq}. This is contained [45, *Proposition* 1.2.6, *page* 10].

Example 3. For $n = 2$ there are four quasi-determinants:

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \qquad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}a_{21}^{-1}a_{22}, \tag{4.5}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}a_{11}^{-1}a_{12}. \tag{4.6}$$

The following lemma is often useful in applications of quasideterminants to determinants [45]. It holds thanks to the Cramer rule for Manin matrices.

Lemma 8.

$$\det \begin{vmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{vmatrix} \tag{4.7}$$

$$= M_{nn}\begin{vmatrix} \boxed{M_{n-1n-1}} & M_{n-1n} \\ M_{nn-1} & M_{nn} \end{vmatrix} \cdots \begin{vmatrix} \boxed{M_{22}} & \dots & M_{2n} \\ \vdots & \ddots & \vdots \\ M_{n2} & \dots & M_{nn} \end{vmatrix} \begin{vmatrix} \boxed{M_{11}} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{vmatrix} \tag{4.8}$$

$$= M_{11}\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & \boxed{M_{22}} \end{vmatrix} \cdots \begin{vmatrix} M_{11} & \dots & M_{1n-1} \\ \vdots & \ddots & \vdots \\ M_{n-11} & \dots & \boxed{M_{n-1n-1}} \end{vmatrix} \begin{vmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & \boxed{M_{nn}} \end{vmatrix}. \tag{4.9}$$

Example 4. For 2×2 Manin matrices:

$$ad - cb = \det^{\mathrm{col}}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d\begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = da - bc. \tag{4.10}$$

4.2. Lagrange–Desnanot–Jacobi–Lewis Carroll formula

Below we present two identities for Manin matrices. The first of the identities is trivial in the commutative case, while the second has a long story: according to D. Bressoud [9], page 111 (remarks after Theorem 3.12) Lagrange found this identity for $n = 3$, Desnanot proved it for $n \leqslant 6$, Jacobi proved the general theorem (see Theorem 3 here), C.L. Dodgson – better known as Lewis Carroll – used it to derive an algorithm for calculating determinants that required only 2×2 determinants ("Dodgson's condensation" method [23]). It is quite surprising how widely such a simple identity appears in various fields of mathematics [117].

The proof of the both identities consists in a small extension and rephrasing of the arguments in the proof of Lemma 1, page 5 of O. Babelon, M. Talon [3].

Proposition 7. *Let M be a Manin matrix and assume that a two sided inverse matrix M^{-1} exists (i.e. $M^{-1}M = MM^{-1} = 1$). Then:*

1. *Column commutativity for M^{-1} holds:*

$$\left(M^{-1}\right)_{ij}\left(M^{-1}\right)_{kj} - \left(M^{-1}\right)_{kj}\left(M^{-1}\right)_{ij} = 0. \tag{4.11}$$

2. *The Lagrange–Desnanot–Jacobi–Lewis Carroll formula holds for Manin matrices, that is*

$$\begin{aligned}&\left(M^{-1}\right)_{ij}\left(M^{-1}\right)_{kl} - \left(M^{-1}\right)_{kj}\left(M^{-1}\right)_{il} \\ &\quad = (-1)^{i+j+k+l}\left(\det^{\mathrm{col}}(M)\right)^{-1}\det^{\mathrm{col}}(\widehat{M}_{jl,ik}), \quad j \neq l,\ i \neq k,\end{aligned} \tag{4.12}$$

   where we use the notation

$$\widehat{M}_{jl,ik} := M_{\text{without } j\text{th and } l\text{th rows; } i\text{th and } k\text{th columns}}$$

   and, in the case $n = 2$ we set by definition $\det^{\mathrm{col}}(\widehat{M}_{12,12}) = 1$.

Remark 18. The standard formulation of the Lagrange–Desnanot–Jacobi–Lewis Carroll formula:

$$\det^{\mathrm{col}}\left(M^{ji}\right)\det^{\mathrm{col}}\left(M^{lk}\right) - \det^{\mathrm{col}}\left(M^{jk}\right)\det^{\mathrm{col}}\left(M^{li}\right) = \left(\det^{\mathrm{col}}(M)\right)\det^{\mathrm{col}}\left(\widehat{M}^{jl;ik}\right), \tag{4.13}$$

can be retrieved (assuming that $\det^{\mathrm{col}}(M)$ commutes with principal minors) multiplying the identity 4.12 by $\det^{\mathrm{col}}(M)^2$ and with the help of the Cramer rule. We recall that we denote by M^{ij} the submatrix without the$i$th row and jth column, and with $\widehat{M}^{jl;ik}$ the $(n-2)\times(n-2)$ submatrix obtained removing the jth and lth rows and the ith and kth columns of M.

Proof. Let us denote by $\Delta = \det^{\mathrm{col}}(M)$, and $\Delta_{ij} = (-1)^{i+j}\det^{\mathrm{col}}(M^{ij})$ the cofactor of M_{ij} in $\det^{\mathrm{col}}(M)$. Consider the Grassmann algebra $\mathbb{C}[\psi_1,\dots,\psi_n]$ (i.e. $\psi_i^2 = 0, \psi_i\psi_j = -\psi_j\psi_i$); let ψ_i commute with M_{pq}, i.e. $\forall i,p,q$: $[\psi_i, M_{pq}] = 0$. Consider the new variables $\tilde\psi_i$:

$$(\tilde\psi_1,\dots,\tilde\psi_n) = (\psi_1,\dots,\psi_n)\begin{pmatrix} M_{11} & \dots & M_{1n} \\ \dots & & \\ M_{n1} & \dots & M_{nn}\end{pmatrix}. \tag{4.14}$$

By Proposition 1,it holds $\tilde\psi_i\tilde\psi_j = -\tilde\psi_j\tilde\psi_i$.

We have the equalities

$$\Delta\cdot\psi_1\wedge\dots\wedge\psi_n \overset{\text{def}}{=} \det^{\mathrm{col}}(M)\psi_1\wedge\dots\wedge\psi_n \overset{\text{lemma 2}}{=} \tilde\psi_1\wedge\dots\wedge\tilde\psi_n, \tag{4.15}$$

$$\Delta_{ij}\psi_1\wedge\dots\wedge\psi_n = (-1)^{j+1}\psi_i\wedge\tilde\psi_1\wedge\dots\wedge\widehat{\tilde\psi_j}\wedge\dots\wedge\tilde\psi_n \quad \text{(here } \tilde\psi_j \text{ is omitted)}, \tag{4.16}$$

$$= \tilde\psi_1\wedge\dots\wedge\overset{j\text{th place}}{\psi_i}\wedge\dots\wedge\tilde\psi_n. \tag{4.17}$$

It is easy to see that the equalities (4.11, 4.12) can be reformulated as follows:

$$\begin{aligned}&\Delta_{ji}M^{-1}_{kl}\psi_1\wedge\dots\wedge\psi_n \\ &\quad = \Delta_{jk}M^{-1}_{il}\psi_1\wedge\dots\wedge\psi_n + \tilde\psi_1\wedge\tilde\psi_2\wedge\dots\wedge\overset{i\text{th place}}{\psi_j}\wedge\dots\wedge\overset{k\text{th place}}{\psi_l}\wedge\dots\wedge\tilde\psi_n.\end{aligned} \tag{4.18}$$

Here we obviously assume $i \neq k$, since for $i = k$ (7) is tautological: $\Delta_{ji}M^{-1}_{il} = \Delta_{ji}M^{-1}_{il}$. Let us prove that the relation (4.18) holds true.

The definition of $\tilde\psi_i$ is $(\tilde\psi_1,\dots,\tilde\psi_m) = (\psi_1,\dots,\psi_n)M$. Multiplying this relation by M^{-1} on the right we get

$$(\tilde\psi_1,\dots,\tilde\psi_n)M^{-1} = (\psi_1,\dots,\psi_n), \quad \text{that is } \sum_v \tilde\psi_v M^{-1}_{vl} = \psi_l.$$

Multiplying $\tilde\psi_1\wedge\tilde\psi_2\wedge\dots\wedge\overset{i\text{th place}}{\psi_j}\wedge\dots\wedge\overset{k\text{th place}}{empty}\wedge\dots\wedge\tilde\psi_n$ and using the Grassmann relations $\tilde\psi_m^2 = 0, i = 1,\dots,n$ we get

$$\left(\tilde\psi_i M^{-1}_{il} + \tilde\psi_k M^{-1}_{kl} - \psi_l\right)\tilde\psi_1\wedge\tilde\psi_2\wedge\dots\wedge\overset{i\text{th place}}{\psi_j}\wedge\dots\wedge\overset{k\text{th place}}{empty}\wedge\dots\wedge\tilde\psi_n = 0.$$

By 4.16 it gives 4.18 and hence Proposition 7 is proved. □

4.3. The inverse of a Manin matrix is again a Manin matrix

Theorem 1. *Let M be a Manin matrix, assume that two sided inverse matrix M^{-1} exists (i.e. $\exists M^{-1}$: $M^{-1}M = MM^{-1} = 1$). Then M^{-1} is again a Manin matrix.*

Remark 19. We will discuss in Section 4.4, that for a reasonable class of rings (which includes main examples) left invertibility of a matrix (not necessarily Manin) implies right invertibility, and hence for Manin matrices invertibility is implied by the invertibility of the determinant. Let us also remark that an analogue of the theorem above holds true for Poisson–Manin matrices (see Section 8.3).

Proof. This statement follows from Proposition 7.[7] Column commutativity for M^{-1} is just formula (4.11) – so it is already established.

For the cross-term relation we notice that

$$[(M^{-1})_{ij}, (M^{-1})_{kl}] = [(M^{-1})_{kj}, (M^{-1})_{il}], \tag{4.19}$$

can be rewritten as:

$$(M^{-1})_{ij}(M^{-1})_{kl} - (M^{-1})_{kj}(M^{-1})_{il} = (M^{-1})_{kl}(M^{-1})_{ij} - (M^{-1})_{il}(M^{-1})_{kj}, \tag{4.20}$$

according to (4.12) both sides of (4.20) are equal to

$$(\det^{\mathrm{col}}(M))^{-1}(-1)^{i+j+k+l}\det^{\mathrm{col}}(M_{\text{without } j\text{th and } l\text{th rows; } i\text{th and } k\text{th columns}}). \tag{4.21}$$

The Theorem 1 is proved. $\square$

Remark 20. We will derive the formula $\det^{\mathrm{col}}(M)^{-1} = (\det M)^{-1}$ for the $n \times n$ case in the next section.

Remark 21. As it is was remarked in [16, Section 4.2.1, page 17] this theorem implies a result by B. Enriquez, V. Rubtsov [27, Theorem 1.1, page 2] and O. Babelon, M. Talon [3, Theorem 2, page 4]. See also [104] for a particular case – about the “commutativity” of quantum Hamiltonians satisfying separation relations, which has important applications in the theory of quantum integrable systems.

4.4. On left and right inverses of a matrix

The main theorems on the inverse of a Manin matrix (Theorem 1) and on the Schur complement (Theorem 2) are formulated under the condition that left and right inverse matrices exist. The lemma below shows that for a reasonable class of rings (which includes main examples) left invertibility of a matrix (not necessarily Manin) implies right invertibility, and hence for Manin matrices invertibility is implied by the invertibility of the determinant.

Lemma 9. *Assume that the ring $\mathcal{K}$ is a noncommutative field (i.e. $r \neq 0 \in \mathcal{K} : \exists\, r^{-1} : r^{-1}r = rr^{-1} = 1$). Then for any matrix $X \in Mat_n(\mathcal{K})$ (for any n), if the left (right) inverse exists, then the right (respectively left) inverse exists also and they coincide.*

If both left and right inverses exist then for any ring associativity guarantees that they coincide: $a_l^{-1} = a_l^{-1}(aa_r^{-1}) = (a_l^{-1}a)a_r^{-1} = a_r^{-1}$.

Proof. Let us prove by induction by the size n of matrix X. For $n = 1$ the lemma is obviously true. Consider general n. At least one element in first column of X is non-zero, otherwise X is not left invertible. Assume it is the element X_{11}, otherwise multiplying by the permutation matrix we put non-zero element to the position (11).

[7] We herewith provide a slightly different proof with respect to that of [16].

Denote the corresponding blocks of the matrix X as B, C, D as follows:

$$X = \begin{pmatrix} X_{11} & B_{1\times n-1} \\ C_{n-1\times 1} & D_{n-1\times n-1} \end{pmatrix}. \tag{4.22}$$

Clearly using the only condition of invertibility of X_{11}, one can write:

$$X \begin{pmatrix} 1 & 0 \\ -CX_{11}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} X_{11} & B \\ 0 & D - CX_{11}^{-1}B \end{pmatrix}. \tag{4.23}$$

Matrix X is invertible from the left by the assumption of the lemma, the other matrix at the left-hand side of the formula above is obviously two-sided invertible, so left-hand side in the formula above is left invertible. So the right-hand side is left invertible and it clearly implies that $n-1 \times n-1$ matrix $D - CX_{11}^{-1}B$ is left invertible. So by the induction it is right invertible also. Clearly:

$$\begin{pmatrix} X_{11} & B \\ 0 & D - CX_{11}^{-1}B \end{pmatrix}^{-1} = \begin{pmatrix} X_{11}^{-1} & -X_{11}^{-1}B(D - CX_{11}^{-1}B)^{-1} \\ 0 & (D - CX_{11}^{-1}B)^{-1} \end{pmatrix}, \tag{4.24}$$

moreover it is two-sided inverse, since element X_{11} is two-sided invertible as any element in $\mathcal{K}$ and $(D - CX_{11}^{-1}B)$ is two-sided invertible by induction. Hence we can present matrix X itself as a product of two-sided invertible matrices:

$$X = \begin{pmatrix} 1 & 0 \\ CX_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} X_{11} & B \\ 0 & D - CX_{11}^{-1}B \end{pmatrix}, \tag{4.25}$$

hence X is two-sided invertible. □

5. Schur's complement and Jacobi's ratio theorem

The main result of this section is a formula for the determinant of a Manin matrix in terms of the determinant of a submatrix and the determinant of the so-called "Schur complement". This theorem is equivalent to the Jacobi's ratio theorem, which expresses a minor of an inverse matrix in term of a complementary minor of matrix itself. The formulations of the both results are exactly the same as in the commutative case. We start with some result on multiplicativity property of the determinant for Manin matrices, which is actually a key point in the proofs of the main theorems. We also show that the so-called Weinstein-Aronszajn and Sylvester formulas hold true for Manin matrices and actually follow from the main theorems.

5.1. Multiplicativity of the determinant for special matrices of block form

The proposition (and lemmas) below is instrumental in the proof of the Schur formula for the determinant of a block matrix to be discussed in the next subsection. It can be proven in a more general form than that strictly needed for the Schur formula, and, in our opinion, is of some interest on its own.

Proposition 8. *Let M be an $n \times n$ Manin matrix, with elements in an associative ring $\mathcal{K}$. Let X be a $k \times (n-k)$ matrix ($k < n$), with arbitrary matrix elements in $\mathcal{K}$. Then*

$$\det^{\mathrm{col}}\left(M \begin{pmatrix} 1_{k\times k} & X_{k\times n-k} \\ 0_{n-k\times k} & 1_{n-k\times n-k} \end{pmatrix}\right) = \det^{\mathrm{col}} M. \tag{5.1}$$

Pay attention that elements of X do not need to commute with elements of M – they are absolutely arbitrary and so the matrix at the left-hand side is not a Manin matrix in general.

Proof. Let us first state the following simple lemmas.

Lemma 10. *Consider elements a_i such that $[a_i, a_j] = 0$. Consider a matrix with the only condition that elements in some columns i and $i+1$ have a form below. Then*

$$\det{}^{\mathrm{col}} \begin{pmatrix} \dots & a_1 & b_1 + a_1 x & \dots \\ \dots & a_2 & b_2 + a_2 x & \dots \\ \dots & \dots & \dots & \dots \\ \dots & a_n & b_n + a_n x & \dots \end{pmatrix} = \det{}^{\mathrm{col}} \begin{pmatrix} \dots & a_1 & b_1 & \dots \\ \dots & a_2 & b_2 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & a_n & b_n & \dots \end{pmatrix}. \tag{5.2}$$

Using the property that one can exchange columns of any Manin matrix, changing only the sign of the column-determinant, column commutativity of the elements of Manin matrices and applying the lemma above, one arrives to the following lemma:

Lemma 11. *Assume M is $n \times k$, $k < n$ Manin matrix, $n \times (n-k-1)$ matrix C is absolutely arbitrary, as well as $n \times 1$ column b and $k \times 1$ column x. Then*

$$\det{}^{\mathrm{col}} \begin{pmatrix} M & b + Mx & C \end{pmatrix} = \det{}^{\mathrm{col}} \begin{pmatrix} M & b & C \end{pmatrix}. \tag{5.3}$$

Where we have used the notation:

$$\begin{pmatrix} M & b + Mx & C \end{pmatrix} = \begin{pmatrix} M_{11} & \dots & M_{1k} & b_1 + \sum_j M_{1j} x_j & C_{11} & \dots & C_{1(n-k-1)} \\ M_{21} & \dots & M_{2k} & b_2 + \sum_j M_{2j} x_j & C_{21} & \dots & C_{2(n-k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ M_{n1} & \dots & M_{nk} & b_n + \sum_j M_{nj} x_j & C_{n1} & \dots & C_{n(n-k-1)} \end{pmatrix}. \tag{5.4}$$

The proofs of the lemmas are trivial.

Let us also remind that without any conditions on the blocks X, Y it is true that:

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & X + Y \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

Let us decompose the matrix X as a sum of its columns:

$$X = \begin{pmatrix} 0 & \dots & 0 & X_{1\,n-k} \\ 0 & \dots & 0 & X_{2\,n-k} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & X_{k\,n-k} \end{pmatrix} + \dots + \begin{pmatrix} X_{11} & 0 & \dots & 0 \\ X_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ X_{k1} & 0 & \dots & 0 \end{pmatrix}. \tag{5.6}$$

Let us denote this decomposition as:

$$X = X_{(n-k)} + \dots + X_{(1)}. \tag{5.7}$$

According to formula (5.5) let us write the corresponding multiplicative decomposition[8]:

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & X_{(n-k)} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & X_{(1)} \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

After these preliminaries the proof of the proposition follows immediately. Observe that we can write

$$M \begin{pmatrix} 1 & X_{(n-k)} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & X_{(n-k-l+1)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M_{11} & \ldots & M_{1(n-l)} & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ M_{n1} & \ldots & M_{n(n-l)} & * & \ldots & * \end{pmatrix}, \tag{5.9}$$

with this meaning that first $n-l$ columns have not been changed, so they satisfy Manin's properties.

Now we can apply Lemma 11:

$$\begin{aligned} &\det^{\mathrm{col}} \left(\begin{pmatrix} M_{11} & \ldots & M_{1n-l} & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ M_{n1} & \ldots & M_{nn-l} & * & \ldots & * \end{pmatrix} \begin{pmatrix} 1 & X_{n-k-l} \\ 0 & 1 \end{pmatrix} \right) \\ &\quad = \det^{\mathrm{col}} \left(\begin{pmatrix} M_{11} & \ldots & M_{1n-l} & * & \ldots & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ M_{n1} & \ldots & M_{nn-l} & * & \ldots & * \end{pmatrix} \right). \end{aligned} \tag{5.10}$$

Applying this equality for $l = 1, \ldots, k$, one finishes the proof:

$$\begin{aligned} &\det^{\mathrm{col}} \left(M \begin{pmatrix} 1 & X_{(n-k)} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & X_{(1)} \\ 0 & 1 \end{pmatrix} \right) = \cdots \\ &\quad = \det^{\mathrm{col}} \left(M \begin{pmatrix} 1 & X_{(n-k)} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & X_{(n-k-l+1)} \\ 0 & 1 \end{pmatrix} \right) = \cdots = \det^{\mathrm{col}} M. \end{aligned} \tag{5.11}$$

The proposition is proven. □

In the same manner on can prove the following

Proposition 9.

- *For M^t a Manin matrix, and an arbitrary block X:*

$$\det^{\mathrm{row}} \left(\begin{pmatrix} 1_{k\times k} & X_{k\times n-k} \\ 0_{n-k\times k} & 1_{n-k\times n-k} \end{pmatrix} M \right) = \det^{\mathrm{row}} M. \tag{5.12}$$

- *Defining* $\det^{\text{col reverse order}} C = \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=n,n-1,n-2,\ldots,1} C_{\sigma(i),i}$, *e.g.* $\det^{\text{col reverse order}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = da - bc$, *and analogously for* $\det^{\text{row reverse order}}$, *it holds*:

$$\det^{\text{col reverse order}} \left(\begin{pmatrix} 1_{k\times k} & 0_{k\times n-k} \\ X^t_{n-k\times k} & 1_{n-k\times n-k} \end{pmatrix} M \right) = \det^{\mathrm{col}} M, \tag{5.13}$$

for M Manin, and

---

[8] In our inductive argument, the chosen order is crucial.

$$\det^{\text{row reverse order}}\left(M\begin{pmatrix}1_{k\times k} & 0_{k\times n-k}\\ X^t_{n-k\times k} & 1_{n-k\times n-k}\end{pmatrix}M\right)=\det^{\text{row}}M, \tag{5.14}$$

for M^t Manin.

5.2. Block matrices, Schur's formula and Jacobi's ratio theorem

Here we prove a formula for the determinant of a block Manin matrix in terms of the determinant of a submatrix and the determinant of the Schur complement. It is one the principal results of the paper. This fact can be equivalently reformulated as Jacobi's ratio theorem which expresses minors of an inverse matrix in terms of minors of the original matrix. The formulations of the theorems are exactly the same as in the commutative case.

In our argument, we prefer to formulate the theorems, their corollaries, and discuss their equivalence first. However, we will provide proofs of each of the two theorems, since these are quite different in flavour.

Theorem 2. *Consider an $n\times n$ Manin matrix M. Let us denote by A, B, C, D its submatrices defined by*

$$M=\begin{pmatrix}A_{k\times k} & B_{k\times n-k}\\ C_{n-k\times k} & D_{n-k\times n-k}\end{pmatrix}, \tag{5.15}$$

where $k<n$. Assume that M, A, D are invertible on both sides, i.e. $\exists M^{-1}$, A^{-1}, $\exists D^{-1}$:

$$M^{-1}M=MM^{-1}=1,\qquad A^{-1}A=AA^{-1}=1,\qquad D^{-1}D=DD^{-1}=1.$$

Then:

1. *It holds*:

$$\det^{\text{col}}(M)=\det^{\text{col}}\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$
$$=\det^{\text{col}}(A)\det^{\text{col}}(D-CA^{-1}B)=\det^{\text{col}}(D)\det^{\text{col}}(A-BD^{-1}C). \tag{5.16}$$

2. *The matrices $(A-BD^{-1}C)$ and $(D-CA^{-1}B)$ are Manin matrices.*

Remark 22. The matrices $(D-CA^{-1}B), (A-BD^{-1}C)$ are called "Schur's complements" respectively of A and D. Also, notice that Schur's formula is exactly that of the usual commutative case.

Proposition 10. *The following more detailed statements hold, as it can be deduced from the proof of Theorem 2.*

$$\det^{\text{col}}(M)=\det^{\text{col}}(A)\det^{\text{col}}\big(D-CA^{-1}B\big),\ \textit{if}\ \exists A^{-1}\colon AA^{-1}=1, \tag{5.17}$$
$$=\det^{\text{col}}(D)\det^{\text{col}}\big(A-BD^{-1}C\big),\ \textit{if}\ \exists D^{-1}\colon DD^{-1}=1, \tag{5.18}$$
$$=\det^{\text{col reverse order}}\big(D-CA^{-1}B\big)\det^{\text{col}}(A),\quad \textit{if}\ \exists A^{-1}\colon A^{-1}A=1, \tag{5.19}$$
$$=\det^{\text{col reverse order}}\big(A-BD^{-1}C\big)\det^{\text{col}}(D),\quad \textit{if}\ \exists D^{-1}\colon D^{-1}D=1. \tag{5.20}$$

We recall that $\det^{\text{col reverse order}}C=\sum_{\sigma\in S_n}(-1)^\sigma\prod_{i=n,n-1,n-2,\dots,1}C_{\sigma(i)i}$, *while* $\det^{\text{col}}$ *is defined via the natural order*: $\prod_{i=1,2,\dots,n}$.

Corollary 2. *Assume M is a Manin matrix. Then*:

$$\det^{\mathrm{col}}(M)\det^{\mathrm{col}}\bigl(M^{-1}\bigr)=1, \quad \textit{if}\ \exists M^{-1}\colon MM^{-1}=1, \tag{5.21}$$

$$\det^{\mathrm{col\ reverse\ order}}\det^{\mathrm{col}}(M)=1, \quad \textit{if}\ \exists M^{-1}\colon M^{-1}M=1. \tag{5.22}$$

And so if M is two-sided invertible, then $\det^{\mathrm{col}}(M)$ *is two-sided invertible and*

$$\det^{\mathrm{col}}(M)^{-1}=\det^{\mathrm{col}}\bigl(M^{-1}\bigr). \tag{5.23}$$

Proof of the Corollary. Assume that $\exists M^{-1}\colon MM^{-1}=1$, consider the $2n\times 2n$ matrix below and apply formula (5.17):

$$(-1)^{n^2}=\det^{\mathrm{col}}\begin{pmatrix} M & 1_{n\times n}\\ 1_{n\times n} & 0_{n\times n}\end{pmatrix}\overset{\text{by (5.16)}}{=}\det^{\mathrm{col}}(M)\det^{\mathrm{col}}\bigl(-M^{-1}\bigr). \tag{5.24}$$

From this one concludes the first statement. Similar arguments prove the second. □

Theorem 2 can be reformulated in the form called "Jacobi's ratio theorem"[9]:

Theorem 3. *Consider a Manin matrix M that admits a left and right inverse M^{-1}. Let* $\det^{\mathrm{col}}(M^{-1}_{I,J})$ *be the minor of M^{-1} indexed by $I=(i_1,\ldots,i_k)$, $J=(j_1,\ldots,j_k)$. Then*

$$\det^{\mathrm{col}}\bigl(M^{-1}_{I,J}\bigr)=(-1)^{\sum_l i_l+\sum_l j_l}\bigl(\det^{\mathrm{col}}(M)\bigr)^{-1}\det^{\mathrm{col}}\bigl(M_{(1,\ldots,n)\setminus J,(1,\ldots,n)\setminus I}\bigr), \tag{5.25}$$

where $\det^{\mathrm{col}}(M_{(1,\ldots,n)\setminus J,(1,\ldots,n)\setminus I})$ *is the minor of the matrix M indexed by the complementary set of indices.*

In other words: *any minor of M^{-1} equals, up to a sign, to the product of* $(\det^{\mathrm{col}}M)^{-1}$ *and the corresponding complementary minor of the* transpose *of M.*[10]

Equivalence of Schur's and Jacobi ratio theorems. The two theorems are actually equivalent. To see this we need to recall the following standard lemma, that holds without any assumptions on commutativity of the matrix elements and matrix blocks involved.

Lemma 12. *Assume that the matrix $\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)$ is invertible from both sides, as well as its submatrices A and D. Then the matrices $(A-BD^{-1}C)$, $(D-CA^{-1}B)$ are also invertible from both sides, and*

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}^{-1}=\begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1}\\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1}\end{pmatrix}. \tag{5.26}$$

Sketch of the proof. The Lemma follows from the factorization formulas below:

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix} 1 & 0\\ CA^{-1} & 1\end{pmatrix}\begin{pmatrix} A & B\\ 0 & D-CA^{-1}B\end{pmatrix} \tag{5.27}$$

$$=\begin{pmatrix} A-BD^{-1}C & B\\ 0 & D\end{pmatrix}\begin{pmatrix} 1 & 0\\ D^{-1}C & 1\end{pmatrix}, \tag{5.28}$$

9 The main formula (5.25) has been also proved for Manin matrices of the form $1-tM$, t is a formal parameter, in the remarkable paper by M. Konvalinka [66] (see Theorem 5.2, page 13). His proof is based on combinatorics.

10 One should pay attention to the "transposition" of indexes: the minor of M^{-1} is indexed by the pair of multi-indices (I,J), while the minor of M is indexed by $((1,\ldots,n)\setminus J,(1,\ldots,n)\setminus I)$.

remarking that

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix}. \tag{5.29}$$

The lemma is proved. □

The equivalence of the two theorems can be given as follows. Consider the set of indexes I, J in Jacobi's ratio theorem to be $I = (1, 2, .., k)$, $J = (1, 2, .., k)$. Denote the corresponding submatrices $M_{I,J} = A$, $M_{(1,2,..,n)\backslash I,(1,2,..,n)\backslash J} = D$, and so on and so forth, i.e. write

$$M = \begin{pmatrix} A_{k\times k} & B_{k\times n-k} \\ C_{n-k\times k} & D_{n-k\times n-k} \end{pmatrix}. \tag{5.30}$$

Assume Jacobi's ratio theorem holds true. This in particular means that M^{-1} is a Manin matrix if M is Manin. According to formula (5.26), $M^{-1}_{I,J} = (A - BD^{-1}C)^{-1}$, so we conclude that $(A - BD^{-1}C)^{-1}$ is a Manin matrix, since it is a submatrix of the Manin matrix M^{-1}. Hence $(A - BD^{-1}C)$ is a Manin matrix as well, by the first claim of Jacobi ratio theorem. Similarly, $(D - CA^{-1}B)$ is a Manin matrix. So the second conclusion of Theorem 2 is derived from Jacobi's ratio theorem.

In order to derive formulas (5.16) in Theorem 2 from Jacobi's ratio theorem we only observe the following. For the case $I = (1, 2, \dots, k)$, $J = (1, 2, dots, k)$ it is true that $M^{-1}_{I,J} = (A - BD^{-1}C)^{-1}$ by (5.26). So Jacobi's ratio formula (5.25) reads, in this case

$$\det{}^{\mathrm{col}}\big((A - BD^{-1}C)^{-1}\big) = \big(\det{}^{\mathrm{col}}(M)\big)^{-1}\det{}^{\mathrm{col}}(D).$$

Since $(A - BD^{-1}C)$ is a Manin matrix, $\det((A - BD^{-1}C)^{-1}) = (\det^{\mathrm{col}}(A - BD^{-1}C))^{-1}$ we arrive at the first claim in Theorem 2.

Thus we have derived Schur's complement Theorem 2 from Jacobi's ratio theorem.

Let us do the converse. Assume that the Schur's complement Theorem 2 is true for a Manin matrix M, with a (right and left) inverse M^{-1}. Construct the $2n \times 2n$ block Manin matrix:

$$M^{ext} = \begin{pmatrix} M & 1_{n\times n} \\ 1_{n\times n} & 0 \end{pmatrix}, \tag{5.31}$$

its Schur's complement $D - CA^{-1}B$ is precisely $-M^{-1}$. So from Theorem 2 we conclude that M^{-1} is a Manin matrix. Also, we can see that $\det^{\mathrm{col}}(M^{-1}) = (\det^{\mathrm{col}}(M))^{-1}$. Indeed by (5.25), $(-1)^{n^2} = \det^{\mathrm{col}}(M^{ext}) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(-M^{-1})$, and, quite obviously, $(-1)^{n^2} = (-1)^n$. So the first part of Jacobi's ratio theorem holds. To derive the second claim for the case $I = (1, 2, \dots, k)$, $J = (1, 2, \dots, k)$ one uses the same arguments as in the discussion after formula (5.30). The statement for arbitrary sets of indexes follows from this special case by changing the order of rows and columns, and taking into account that changing the order of rows in M implies change of order of columns in M^{-1}, and vice versa. The equivalence of Schur's and Jacobi ratio theorems is thus established.

5.2.1. Proof of the Schur's complement theorem

Proof 1. Let us prove Theorem 2. To this end, we first prove formulas (5.17) and (5.18) which are more refined statements of the first claim (formula 5.16) in Theorem 2. The proof uses the same idea as in the commutative case and Proposition 8.

Let us consider the standard decomposition[11]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix}. \tag{5.32}$$

By Proposition 8 one gets the first equality: $\det^{\mathrm{col}} M = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(D - CA^{-1}B)$. This is desired formula (5.17).

To prove the equality (5.18), i.e. $\det^{\mathrm{col}}(M) = \det^{\mathrm{col}}(D)\det^{\mathrm{col}}(A - BD^{-1}C)$, one observes that since M is Manin

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{nk} \det \begin{pmatrix} B & A \\ D & C \end{pmatrix} \tag{5.33}$$

by column transposition.

Now we can change the order of rows, (which is possible for the column-determinant of any matrix) to get

$$\det \begin{pmatrix} B & A \\ D & C \end{pmatrix} = (-1)^{nk} \det \begin{pmatrix} D & C \\ B & A \end{pmatrix}. \tag{5.34}$$

So we one gets:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} D & C \\ B & A \end{pmatrix} = \det^{\mathrm{col}}(D)\det^{\mathrm{col}}\bigl(A - BD^{-1}C\bigr). \tag{5.35}$$

The last equality holds to a factorization analogous to the one of Eq. (5.32).

To prove the remaining formulas (5.19), (5.20) one uses the same arguments as above for the decomposition:

$$\begin{pmatrix} 1 & 0 \\ -CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}, \tag{5.36}$$

where now A^{-1} is left inverse to A.

We are left with proving that the Schur's complements $(A - BD^{-1}C)$, $(D - CA^{-1}B)$ are Manin matrices. To do this we recall Lemma 12 page 269:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{5.37}$$

From Theorem 1 we know that M^{-1} is a Manin matrix if M is a Manin matrix and M is two sided invertible. Trivially any submatrix of a Manin matrix is again a Manin matrix. So we conclude that $(A - BD^{-1}C)^{-1}$ and $(D - CA^{-1}B)^{-1}$ are Manin matrices. Applying Theorem 1 again we conclude that $(A - BD^{-1}C)$ and $(D - CA^{-1}B)$ are Manin matrices. □

[11] Observe that there is no need in any commutativity constrains, but only existence of the (right) inverse of the upper left block A.

5.2.2. Proof 2. (Proof of the Jacobi ratio theorem via quasideterminants)

Let us now give a proof of Jacobi's ratio Theorem 3. Our proof is quite simple and non-computational, the arguments being borrowed from: [69, Theorem 3.2, page 17], and the theory of quasideterminants [45]. Roughly speaking it goes as follows: one represents the determinants appearing in Jacobi's formula (3) as a product of quasiminors by Lemma 8 (essentially by Cramer's rule). Using Lemma 13 (which we prove below), the result then follows from Cramer's rule. Of course, we consider (and make use of) the fact that M^{-1} is also a Manin matrix as already established (Theorem 1).

Proof. First, let us recall the following lemma, which is called noncommutative Jacobi's ratio theorem ([41], [43, Theorem 1.3.3, page 8], [69, Theorem 2.4, page 8]) or inversion law for quasiminors [45, Theorem 1.5.4, page 19]:

Lemma 13. *For an arbitrary invertible matrix A (not necessarily a Manin matrix), it is true that the (ji)th quasiminor of A^{-1} is the inverse of the "almost" complementary (ij)th quasiminor of M. Here by almost complementary we mean the complementary united with the ith row and jth column*

$$\left|A^{-1}_{P,Q}\right|_{ij} = |A_{\{1\ldots n\}-P\cup i,\{1\ldots n\}-Q\cup j}|^{-1}_{ji}, \tag{5.38}$$

where $A_{I,J}$ is a submatrix indexed by index sets I, J.

In particular for $P = Q = 1,\ldots,k$ and $i = j = k$:

$$\begin{vmatrix} A^{-1}_{11} & \ldots & A^{-1}_{1k} \\ \ldots & \ldots & \ldots \\ A^{-1}_{k1} & \ldots & \boxed{A^{-1}_{kk}} \end{vmatrix}_{kk} = \begin{vmatrix} \boxed{A_{kk}} & \ldots & A_{kn} \\ \ldots & \ldots & \ldots \\ A_{nk} & \ldots & A_{nn} \end{vmatrix}^{-1}_{kk}. \tag{5.39}$$

The proof of this Lemma quite readily follows from Lemma 12.

Coming back to the Jacobi ratio theorem, we need to prove:

$$\det{}^{\mathrm{col}}\bigl(M^{-1}_{I,J}\bigr) = (-1)^{\sum_l i_l + \sum_l j_l}\bigl(\det{}^{\mathrm{col}}(M)\bigr)^{-1}\det{}^{\mathrm{col}}\bigl(M_{(1,\ldots,n)-J,(1,\ldots,n)-I}\bigr). \tag{5.40}$$

Possibly changing the order of rows and columns (which is possible for M and M^{-1} since both are Manin matrices (Theorem 1) we reduce this identity to the case $I = J = \{1,\ldots,k\}$:

$$\det{}^{\mathrm{col}}\bigl(M^{-1}_{k}\bigr) = \bigl(\det{}^{\mathrm{col}}(M)\bigr)^{-1}\det{}^{\mathrm{col}}(M_{\backslash k}), \tag{5.41}$$

where M^{-1}_k is the submatrix of M^{-1} made of the first k rows and columns, and $M_{\backslash k}$ is the submatrix of M made of the last $(n-k)$ rows and columns. One should pay attention to the fact that changing the order of rows implies changing the order of columns of the inverse matrix; this implies a possible sign factor and explains the signs and transposition of index sets in formula (5.40).

By Theorem 1 M^{-1} is also a Manin matrix, so we can use Cramer's rule 6 for it, as well as for M itself.

Let us multiply $\det^{\mathrm{col}}(M^{-1}_k) =$ by $1 = \prod_{i=1,\ldots,k-1}\det^{\mathrm{col}}(M^{-1}_i)/\det^{\mathrm{col}}(M^{-1}_i)$ to get

$$\det{}^{\mathrm{col}}\bigl(M^{-1}_k\bigr) \tag{5.42}$$

$$= M^{-1}_{11}\bigl(M^{-1}_{11}\det{}^{\mathrm{col}}\bigl(M^{-1}_2\bigr)\bigr)\bigl(\det{}^{\mathrm{col}}\bigl(M^{-1}_2\bigr)^{-1}\det{}^{\mathrm{col}}\bigl(M^{-1}_3\bigr)\bigr)\ldots\bigl(\det{}^{\mathrm{col}}\bigl(M^{-1}_{k-1}\bigr)^{-1}\det{}^{\mathrm{col}}\bigl(M^{-1}_k\bigr)\bigr) \tag{5.43}$$

$$\text{by Cramer's rule:}\quad \bigl|M^{-1}_k\bigr|_{kk} = \det{}^{\mathrm{col}}\bigl(M^{-1}_{k-1}\bigr)^{-1}\det{}^{\mathrm{col}}\bigl(M^{-1}_k\bigr) \tag{5.44}$$

$$= M_{11}^{-1}\left|M_2^{-1}\right|_{22}\left|M_3^{-1}\right|_{33}\dots\left|M_k^{-1}\right|_{kk} \quad \text{by Lemma 13}$$
$$= |M|_{11}^{-1}|M_{\backslash 1}|_{22}^{-1}|M_{\backslash 2}|_{33}^{-1}\dots|M_{\backslash k-1}|_{kk}^{-1} \quad \text{by Cramer's rule for } M$$
$$= \left(\det^{\mathrm{col}}(M)^{-1}\det^{\mathrm{col}}(M_{\backslash 1})\right)\left(\det^{\mathrm{col}}(M_{\backslash 1})^{-1}\det^{\mathrm{col}}(M_{\backslash 2})\right)\dots\left(\det^{\mathrm{col}}(M_{\backslash k-1})^{-1}\det^{\mathrm{col}}(M_{\backslash k})\right) \tag{5.45}$$

by chain cancellation:

$$= \det^{\mathrm{col}}(M)^{-1}\det^{\mathrm{col}}(M_{\backslash k}). \tag{5.46}$$

Formula (5.41) is proved, hence the theorem is proved. □

5.3. The Weinstein–Aronszajn formula

Proposition 11. *Let A, B be $n \times k$ and $k \times n$ Manin matrices with pairwise commuting elements*: $\forall i, j, k, l$: $[A_{ij}, B_{kl}] = 0$. *Then*

$$\det^{\mathrm{col}}(1_{n\times n} - AB) = \det^{\mathrm{col}}(1_{k\times k} - BA). \tag{5.47}$$

Proof. Consider the following matrix:

$$\begin{pmatrix} 1_{k\times k} & B \\ A & 1_{n\times n} \end{pmatrix}. \tag{5.48}$$

It is clearly a Manin matrix. Applying formula (5.16) one obtains the result. □

Remark 23. The name Weinstein–Aronszajn formula comes from [62, Chapter 4, Section 6], in the analysis of finite rank perturbation of operators in (infinite-dimensional) Hilbert spaces. The formula is used in the theory of integrable systems (see, e.g. H. Flaschka, J. Millson [34, Section 6.1, page 23], K. Takasaki [109, page 10]), as follows. One considers the matrix

$$M_n := \mathbf{1}_{n\times n} - \sum_{\alpha=1}^{k} x_\alpha \otimes y_\alpha, \quad x_\alpha, y_\alpha \in \mathbb{C}^n.$$

It can be considered as a perturbation of the identity operator by means of the k rank 1 operators $x_\alpha \otimes y_\alpha, \alpha = 1, \dots, k$. The Weinstein–Aronszajn formula reads

$$\det^{\mathrm{col}}(M_n) = \det^{\mathrm{col}}(\mathbf{1}_{k\times k} - S_k),$$

where S_k is the $k \times k$ matrix whose element $S_{\alpha,\beta}$ is the "scalar" product (x_α, y_β).

To get this form from our result, one simply sets B to be the matrix whose αth row is x_α, and A is the matrix whose βth column if y_β in the expression (5.48).

Thus Proposition 11 holds for this case of Manin matrices as well.

5.4. Sylvester's determinantal identity

Sylvester's identity is a classical determinantal identity (see, e.g. [45, Theorem 1.5.3, page 18]). Using combinatorial methods it has been generalized for Manin matrices of the form $1+M$ and their's q-analogs by M. Konvalinka [67]. Here (following the classical paper by E.H. Bareiss [5]) we show that the identity easily follows from the Schur formula above (Theorem 2).

Let us first recall the commutative case:

Theorem 4 *(Commutative Sylvester's identity). Let A be a matrix $(a_{ij})_{m\times m}$; take $n<i,j\leqslant m$; denote:*

$$A_0=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix},\qquad a_{i*}=(\,a_{i1}\quad a_{i2}\quad \cdots\quad a_{in}\,),\qquad a_{*j}=\begin{pmatrix} a_{1j}\\ a_{2j}\\ \vdots\\ a_{nj}\end{pmatrix}. \tag{5.49}$$

Define the $(m-n)\times(m-n)$ matrix B as follows:

$$B_{ij}=\det\begin{pmatrix} A_0 & a_{*j}\\ a_{i*} & a_{ij}\end{pmatrix},\qquad B=(B_{ij})_{n+1\leqslant i,j\leqslant m}. \tag{5.50}$$

Then

$$\det B=\det A\cdot(\det A_0)^{m-n-1}. \tag{5.51}$$

Theorem 5 *(Sylvester's identity for Manin matrices). Let M be $m\times m$ a Manin matrix with right and left inverse; take $n<i,j\leqslant m$ and denote:*

$$M_0=\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n}\\ M_{21} & M_{22} & \cdots & M_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ M_{n1} & M_{n2} & \cdots & M_{nn}\end{pmatrix},\qquad M_{i*}=(\,M_{i1}\quad M_{i2}\quad \cdots\quad M_{in}\,),\qquad M_{*j}=\begin{pmatrix} M_{1j}\\ M_{2j}\\ \vdots\\ M_{nj}\end{pmatrix}. \tag{5.52}$$

Define the $(m-n)\times(m-n)$ matrix B as follows:

$$B_{ij}=\bigl(\det^{\mathrm{col}}(M_0)\bigr)^{-1}\cdot\det\begin{pmatrix} M_0 & M_{*j}\\ M_{i*} & M_{ij}\end{pmatrix},\qquad B=(B_{ij})_{n+1\leqslant i,j\leqslant m}. \tag{5.53}$$

Then the matrix B is a Manin matrix and

$$\det B=(\det M_0)^{-1}\cdot\det M. \tag{5.54}$$

Remark 24. Formula (5.54) reduces to (5.51) in the commutative case. In the noncommutative case, the Sylvester identity holds in the form (5.54).

Proof. Once chosen M_0, we consider the resulting block decomposition of M,

$$M=\begin{pmatrix} M_0 & M_1\\ M_2 & M_3\end{pmatrix}. \tag{5.55}$$

The key observation is the following:

Lemma 14. *The matrix B defined by* (5.53) *equals to the Schur complement matrix:* $M_0 - M_2(M_3)^{-1}M_1$.

To see this, we need to use Schur complement Theorem (Theorem 2) again:

$$B_{ij} = \left(\det^{\mathrm{col}}(M_0)\right)^{-1} \cdot \det\begin{pmatrix} M_0 & M_{*j} \\ M_{i*} & M_{ij}\end{pmatrix}$$

$$= \left(\det^{\mathrm{col}}(M_0)\right)^{-1} \cdot \left(\left(\det^{\mathrm{col}}(M_0)\right)\left(M_{ij} - M_{i*}M_0^{-1}M_{*j}\right)\right) \tag{5.56}$$

$$= \left(M_{ij} - M_{i*}M_0^{-1}M_{*j}\right) = \left(M_0 - M_2(M_3)^{-1}M_1\right)_{ij}. \tag{5.57}$$

In particular, we used the Schur formula $\det^{\mathrm{col}}(M) = \det^{\mathrm{col}}(A)\det^{\mathrm{col}}(D - CA^{-1}B)$ for blocks $M_0 = A$, $M_{*j} = B$, $M_{i*} = C$, $M_{ij} = D$, the last being a 1×1 matrix. So the lemma is proved.

The theorem follows from the lemma immediately; indeed, $B = (M_0 - M_2(M_3)^{-1}M_1)$ is a Manin matrix, since a Schur complement is a Manin matrix by Theorem 2. $\det B = (\det M_0)^{-1} \cdot \det M$ follows from the formula for determinant of the Schur complements. □

Remark 25 *(Bibliographical notes).* Sylvester's identity for quasi-determinants has been found in [41] (see also I. Gelfand, S. Gelfand, V. Retakh, R. Wilson [45, Theorem 1.5.2, page 18]). The generalization to quantum matrices in D. Krob, B. Leclerc [69, Theorem 3.5, page 18]. Using combinatorial methods it has been generalized for Manin matrices of the form $1 + M$ and their's q-analogs in M. Konvalinka [67]. The identity for Yangians and q-affine algebras can be found in Section 2.12, page 18, A.I. Molev [79], Section 1.12 [81] and Section 3, page 8, M.J. Hopkins, A.I. Molev [55], respectively; for twisted Yangians in Section 3, page 15, A.I. Molev [80], Section 2.14 [81]. These facts are used in the so-called centralizer construction for the corresponding algebras and have some other applications. Commutative version of the identity is discussed in various texts (e.g. [85]), we followed E.H. Bareiss [5].

5.5. Application to numeric matrices

Let us discuss a corollary on a calculation of the usual determinants of specific numeric matrices, which in principle might provide faster algorithm for calculating the determinant of such matrices.

Proposition 12. *Consider $nm \times nm$ matrix $\widetilde{M}$ with elements in a commutative ring K. Divide it into $m \times m$ square blocks. Denote by M an $n \times n$ matrix over $Mat_m(K)$, which matrix elements are corresponding blocks of $\widetilde{M}$. Assume M is an $n \times n$ Manin matrix over $Mat_m(K)$. Then the determinant of $\widetilde{M}$ can be calculated in two steps: first one calculates $n \times n$-column-determinant of a corresponding $n \times n$-matrix M over $Mat_m(K)$, this determinant is itself an $m \times m$ matrix B over K, second one calculates the determinant of B in the usual sense*:

$$\det_{nm\times nm}(\widetilde{M}) = \det_{m\times m}\left(\det^{\mathrm{col}}_{n\times n}(M)\right), \tag{5.58}$$

we denoted by $\det_{r\times r}$ *determinants of $r \times r$-matrices.*

Clearly such a formula is not true in general without assuming that M is a Manin matrix.

Example 5. Let $n = 2$, so we consider $2m \times 2m$ matrix over K which is divided into 4 blocks of size $m \times m$, and it is a 2×2 Manin matrix over $Mat_m(K)$ (i.e. $[a, c] = [b, d] = 0$, $[a, d] = [c, b]$). Then:

$$\det_{2m\times 2m}\begin{pmatrix} a_{m\times m} & b_{m\times m} \\ c_{m\times m} & d_{m\times m}\end{pmatrix} = \det_{m\times m}(a_{m\times m}d_{m\times m} - c_{m\times m}b_{m\times m}). \tag{5.59}$$

Proof. Let us fix m and prove by induction in n. For $n=1$ the statement is a tautology.

Consider general n. Let us assume that $\det_{nm\times nm}(\widetilde{M})\neq 0$ and so it is two-sided invertible, otherwise it is quite easy to see that the proposition is true.

Let us denote by B, C, D blocks of the matrix M:

$$M=\begin{pmatrix} M_{11} & \dots & M_{1n}\\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nn}\end{pmatrix}=\begin{pmatrix} M_{11} & B\\ C & D\end{pmatrix}, \tag{5.60}$$

here M_{ij} are themselves $m\times m$ matrices over K. We may assume that M_{11} is invertible, otherwise one should make a permutation of rows or columns. The Schur complement formula (5.16) can be applied for matrices over commutative rings:

$$\det_{nm\times nm}(\widetilde{M})=\det_{m\times m}(M_{11})\det_{(n-1)m\times(n-1)m}\bigl(D-CM_{11}^{-1}B\bigr), \tag{5.61}$$

on the other hand M is a Manin matrix over $Mat_m(K)$, so we can use the Schur complement formula in the following way:

$$\det^{\mathrm{col}}_{n\times n}(M)=M_{11}\det^{\mathrm{col}}_{(n-1)\times(n-1)}\bigl(D-CM_{11}^{-1}B\bigr), \tag{5.62}$$

so

$$\det_{m\times m}\bigl(\det^{\mathrm{col}}_{n\times n}(M)\bigr)=\det_{m\times m}(M_{11})\bigl(\det^{\mathrm{col}}_{(n-1)\times(n-1)}\bigl(D-CM_{11}^{-1}B\bigr)\bigr)= \tag{5.63}$$

$$=\det_{m\times m}(M_{11})\bigl(\det_{m\times m}\bigl(\det^{\mathrm{col}}_{(n-1)\times(n-1)}\bigl(D-CM_{11}^{-1}B\bigr)\bigr)\bigr) \tag{5.64}$$

by Theorem 2 on Schur complements for Manin matrices one knows that $(D-CM_{11}^{-1}B)$ is a Manin matrix, so by the induction hypothesis:

$$=\det_{m\times m}(M_{11})\bigl(\det_{(n-1)m\times(n-1)m}\bigl(D-CM_{11}^{-1}B\bigr)\bigr). \tag{5.65}$$

This coincides with (5.61), so the proposition is proved. □

Remark 26. Taking $R=Mat_m(K)$ and considering examples 3.37 one obtains examples of matrices of the form considered in the proposition. It is however still unclear to us whether such block-Manin matrices may appear in practical numerical applications.

6. Cauchy–Binet formulae and Capelli-type identities

We have already discussed (Proposition 4) that $\det^{\mathrm{col}}(MY)=\det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y)$ if $[M_{ij},Y_{kl}]=0$ and M is a Manin matrix, and actually one can prove in the same way, the Cauchy–Binet formulae: $\det^{\mathrm{col}}((MY)_{IJ})=\sum_L\det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y_{LJ})$. In a recent remarkable paper S. Caracciolo, A. Sportiello, A. Sokal [12] found an unexpected noncommutative analogue of the Cauchy–Binet formulae for Manin matrices, for the case $[M_{ij},Y_{kl}]\neq 0$ – but subjected to obey certain conditions. Remark that the left-hand side of their formulae contains a correction: $\det^{\mathrm{col}}((MY)_{IJ}+H)$. As a particular case of their identity one obtains the classical Capelli and related identities. This section is based on our interpretation of [12]; we shall give a generalization of their results and provide different and possibly (in our opinion) more transparent proofs. Also, we obtain similar formulas for permanents. Our main tool is use of the Grassmann algebra for calculations with the determinants and respectively polynomial algebra for the case of the permanents. The condition found in [12] has a natural reformulation in terms of these algebras and implies that certain expressions (anti)commute, as if it would be a commutative case (see Lemmas 19, 20 and Lemma 23).

Here and below we use the following quite standard notations.

Notation 1. Let A be an $n \times m$ matrix. Consider multi-indexes $I = (i_1, \ldots, i_{r_1})$, $J = (j_1, \ldots, j_{r_2})$. We denote by A_{IJ} the following $r_1 \times r_2$ matrix:

$$(A_{IJ})_{ab} = \begin{cases} A_{i_a j_b}, & i_a \leqslant n,\ j_b \leqslant m, \\ 0, & i_a > n \text{ or } j_b > m. \end{cases} \tag{6.1}$$

Note that I, J are not assumed to be ordered and any number α may occur several times in the sequences I, J.

Example 6. Even if A is 1×1 matrix, we can construct $2 \times 2, \ldots$ matrices from it:

$$A_{(11)(11)} = \begin{pmatrix} A_{11} & A_{11} \\ A_{11} & A_{11} \end{pmatrix}, \qquad A_{(12)(11)} = \begin{pmatrix} A_{11} & A_{11} \\ 0 & 0 \end{pmatrix}. \tag{6.2}$$

Notation 2. Given the elements $\psi_1, \ldots, \psi_n$ and an $n \times m$ matrix A, for brevity we denote by ψ_i^A the element $\sum_{k=1,\ldots,n} \psi_k A_{ki}$ i.e. just the application of the matrix A to the row-vector $(\psi_1, \ldots, \psi_n)$:

$$\left(\psi_1^A, \ldots, \psi_m^A\right) = (\psi_1, \ldots, \psi_n)A. \tag{6.3}$$

Viceversa, if we have some elements $\psi_1, \ldots, \psi_n$ and $\psi_1^A, \ldots, \psi_m^A$ we denote by A a matrix (if it exists) such that $(\psi_1^A, \ldots, \psi_m^A) = (\psi_1, \ldots, \psi_n)A$.

6.1. Grassmann algebra condition for Cauchy–Binet formulae

Here we prove the Cauchy–Binet formulae under certain conditions on our matrices; some of the results of [12] will be obtained as particular cases thereof.

Let M be an $n \times m$ matrix and Y an $m \times s$ matrix with elements in the ring $\mathcal{K}$. Consider the Grassmann algebra $\Lambda[\psi_1, \ldots, \psi_n]$ (i.e. $\psi_i^2 = 0$, $\psi_i\psi_j = -\psi_j\psi_i$), and denote by $\psi_i^M = \sum_k \psi_k M_{ki} \in \mathcal{K} \otimes \Lambda[\psi_1, \ldots, \psi_n]$.

We will be interested in the situation where the matrices M, Y satisfy the following two conditions.

Condition 1.

$$\forall p, j\ \exists \psi_j^Q \in \mathcal{K} \otimes \Lambda[\psi_1, \ldots, \psi_n] \quad \text{such that} \quad \sum_{l=1,\ldots,m} \psi_l^M \left[Y_{lj}, \psi_p^M\right] = \psi_p^M \psi_j^Q. \tag{6.4}$$

Notice that we require that ψ_j^Q be independent of the index p.

Here we denote by Q the $n \times s$ matrix associated with the elements ψ_i^Q as follows:

$$\left(\psi_1^Q, \ldots, \psi_s^Q\right) = (\psi_1, \ldots, \psi_n)Q. \tag{6.5}$$

Condition 2. ψ_i^M and ψ_j^Q anticommute:

$$\forall i, j, \quad \psi_i^M \psi_j^Q = -\psi_j^Q \psi_i^M. \tag{6.6}$$

Theorem 6. *Assume that M is an $n \times m$ Manin matrix, and let Y be an arbitrary $m \times s$ matrix (i.e. not necessarily a Manin matrix). Suppose that the matrices M and Y satisfy Conditions* 1 *and* 2 ((6.4), (6.6)) *above. If $n = m = s$, then*:

$$\det^{\mathrm{col}}\big(MY + Q\ \mathrm{diag}(n-1, n-2, \ldots, 1, 0)\big) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y), \tag{6.7}$$

where Q is a matrix corresponding to the elements ψ_i^Q according to formula (6.5) *and the elements ψ_i^Q arise in the condition 1 above; by* $\mathrm{diag}(a_1, a_2, \ldots)$ *we denote the matrix with a_i on the diagonal and* 0*'s elsewhere.*

More generally the following Cauchy–Binet formulae hold. Let $I = (i_1 < i_2 < \cdots < i_r)$, $J = (j_1, \ldots, j_r)$[12]*; be two multi-indexes with $i_a \leqslant n$, $j_a \leqslant s$, $r \leqslant n, s$. Then*

$$\det^{\mathrm{col}}\big((MY)_{IJ} + Q_{IJ}\ \mathrm{diag}(r-1, r-2, \ldots, 1, 0)\big) = \sum_{L=(l_1<l_2<l_3<\cdots<l_r\leqslant m)} \det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y_{LJ}), \tag{6.8}$$

here Q_{IJ} is the matrix such that $(Q_{IJ})_{ab} = Q_{i_a j_b}$, according to the Notation 1.

Before giving the proof of this formula, let us present some of its corollaries.

Corollary 3. *Consider the case $m < n, s$,*

$$M = \begin{pmatrix} M_{11} & \ldots & M_{1m} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ M_{n1} & \ldots & M_{nm} \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_{11} & \ldots & \ldots & \ldots & Y_{1s} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ Y_{m1} & \ldots & \ldots & \ldots & Y_{ms} \end{pmatrix}, \tag{6.9}$$

then for any $r > m$:

$$\det^{\mathrm{col}}\big((MY)_{IJ} + Q_{IJ}\ \mathrm{diag}(r-1, r-2, \ldots, 1, 0)\big) = 0. \tag{6.10}$$

Indeed, for $r > m$ there is no such L that $(l_1 < l_2 < l_3 < \cdots < l_r \leqslant m)$, and so there is no terms in the sum at the right-hand side. It is the same as in the commutative case, where rank of MY does not exceed m and so all minors of size $r > m$ are zeros.

Corollary 4. *Assume that the matrix Y is also a Manin matrix, and consider the matrix $(MY)^\sigma$ obtained as an arbitrary permutation σ of columns of MY. Then*

$$\begin{aligned} &\det^{\mathrm{col}}\big((MY)^\sigma_{IJ} + Q^\sigma_{IJ}\ \mathrm{diag}(r-1, r-2, \ldots, 1, 0)\big) \\ &\quad = (-1)^{sgn(\sigma)}\det^{\mathrm{col}}\big((MY)_{IJ} + Q_{IJ}\ \mathrm{diag}(r-1, r-2, \ldots, 1, 0)\big). \end{aligned} \tag{6.11}$$

Indeed, it is easy to see that matrices M and Y^σ satisfy Conditions 1 and 2 with the matrix Q^σ and $(MY)^\sigma = M(Y)^\sigma$, so using formulae (6.8) in the main theorem we obtain from the left-hand side of (6.11) sum of terms $\det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y^\sigma_{LJ})$, and from the right-hand side we obtain $\det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y_{LJ})$, since Y is a Manin matrix, we see that they are equal up to $(-1)^{sgn(\sigma)}$.

[12] Conditions $j_a < j_b$, $j_a \neq j_b$ are *not* required.

To prove the theorem, we will make use of the following two Lemmas. The first one extends formulae that are well known and obvious in the commutative case. However we prefer to explicitly notice how they extend to the noncommutative case in a straightforward way.

Lemma 15. *Consider Grassmann variables ψ_i and an $n \times m$ matrix A; let $J = (j_1, j_2, \dots, j_r)$ be an arbitrary multi-index (i.e. it is* not *assumed that $j_a \neq j_b$, nor $j_a < j_b$), and assume that ψ_i commute with A_{kl}. Then*

$$\psi^A_{j_1}\psi^A_{j_2}\dots\psi^A_{j_r} = \sum_{L=(l_1<l_2<l_3<\cdots<l_r\leqslant n)} \psi_{l_1}\psi_{l_2}\dots\psi_{l_r}\det{}^{\mathrm{col}}(A_{LJ}). \tag{6.12}$$

Dropping the assumption $[\psi_i, A_{kl}] = 0$ we can write nonetheless the equality:

$$\sum_{I=(i_1,\dots,i_r:\ 1\leqslant i_a\leqslant m)} \psi_{i_1}\psi_{i_2}\dots\psi_{i_r}A_{i_1j_1}A_{i_2j_2}\dots A_{i_rj_r} = \sum_{L=(l_1<l_2<l_3<\cdots<l_r\leqslant n)} \psi_{l_1}\psi_{l_2}\dots\psi_{l_r}\det{}^{\mathrm{col}}(A_{LJ}). \tag{6.13}$$

The expansion of the column determinant with respect to the first column implies the following:

$$\sum_{l_1=1,\dots,n}\psi_{l_1}\sum_{L^-=(l_2<l_3<\cdots<l_r\leqslant n)} \psi_{l_2}\dots\psi_{l_r}A_{l_1j_1}\det{}^{\mathrm{col}}(A_{L^-J^-}) = \sum_{L=(l_1<l_2<l_3<\cdots<l_r\leqslant n)} \psi_{l_1}\psi_{l_2}\dots\psi_{l_r}\det{}^{\mathrm{col}}(A_{LJ}), \tag{6.14}$$

where $J^- = (j_2, j_3, \dots, j_r)$.

The second lemma is somehow less obvious.

Lemma 16. *Conditions* 1, *Eq.* (6.4), *and* 2, *Eq.* (6.6) *guarantee that*

$$\sum_{l_1}\psi^M_{l_1}\bigl[Y_{l_1j_1}, \psi^M_{l_2}\dots\psi^M_{l_r}\bigr] + (r-1)\psi^Q_{j_1}\psi^M_{l_2}\dots\psi^M_{l_r} = 0. \tag{6.15}$$

Proof. Let us transform the first term by the Leibniz rule, to get

$$\sum_{l_1}\psi^M_{l_1}\bigl[Y_{l_1j_1}, \psi^M_{l_2}\dots\psi^M_{l_r}\bigr] = \sum_{l_1}\psi^M_{l_1}\sum_{p=2,\dots,r}\psi^M_{l_2}\dots\bigl[Y_{l_1j_1}, \psi^M_{l_p}\bigr]\dots\psi^M_{l_r} \tag{6.16}$$

using Manin's property (Proposition 1) we know that ψ^M_i anticommute among themselves so we can move $\psi^M_{l_1}$ in front of $[Y_{l_1j_1}, \psi^M_{l_p}]$ (getting the sign factor $(-1)^{p-2}$). Using condition (6.4), that is, $\sum_l \psi^M_l[Y_{lj}, \psi^M_p] = \psi^M_p\psi^Q_j$, we get that the right-hand side of Eq. (6.16) equals

$$(-1)^{p-2}\sum_{p=2,\dots,r}\psi^M_{l_2}\dots\psi^M_{l_p}\psi^Q_{j_1}\dots\psi^M_{l_r}. \tag{6.17}$$

Now, using anticommutativity of ψ^M and ψ^Q (condition (6.6)), we put $\psi^Q_{l_p}$ in front of the expression and get the sign factor $(-1) = (-1)^{p-2}(-1)^{p-1}$. Thus we see that

$$\sum_{l_1} \psi_{l_1}^M \big[Y_{l_1 j_1}, \psi_{l_2}^M \dots \psi_{l_r}^M\big] = (-1)\psi_{j_1}^Q \sum_{p=2,\dots,r} \psi_{l_2}^M \dots \psi_{l_p}^M \dots \psi_{l_r}^M. \tag{6.18}$$

In the sum $\sum_{p=2,\dots,r}$ we see all the terms are identically the same, so we have that the right-hand side of the above equation reduces to:

$$= (-1)(r-1)\psi_{j_1}^Q \psi_{l_2}^M \dots \psi_{l_r}^M, \tag{6.19}$$

which is exactly the right-hand side in the lemma. The lemma is proved. □

Let us finally turn to the proof of Theorem 6.

Proof. [13] The equality (6.8) in the theorem can be reformulated (with the help of (6.12)) in terms of the Grassmann algebra as follows:

$$\big(\psi_{j_1}^{MY} + (r-1)\psi_{j_1}^Q\big)\big(\psi_{j_2}^{MY} + (r-2)\psi_{j_2}^Q\big)\dots\big(\psi_{j_r}^{MY}\big) \tag{6.20}$$

$$= \sum_{L=(l_1<l_2<l_3<\dots<l_r)} \sum_{I=(i_1<i_2<i_3<\dots<i_r)} \psi_{i_1}\dots\psi_{i_r} \det{}^{\mathrm{col}}(M_{IL}) \det{}^{\mathrm{col}}(Y_{LJ}). \tag{6.21}$$

By (6.12) the right-hand side of equality (6.20) can be also rewritten as:

$$\sum_{L=(l_1<l_2<l_3<\dots<l_r)} \psi_{l_1}^M \dots \psi_{l_r}^M \det{}^{\mathrm{col}}(Y_{LJ}). \tag{6.22}$$

Transform the left-hand side of (6.20) using induction and (6.22):

$$\big(\psi_{j_1}^{MY} + (r-1)\psi_{j_1}^Q\big)\big(\psi_{j_2}^{MY} + \psi_{j_2}^Q\big)\dots\big(\psi_{j_r}^{MY} + \psi_{j_r}^Q\big) \tag{6.23}$$

$$= \big(\psi_{j_1}^{MY} + (r-1)\psi_{j_1}^Q\big)\left(\sum_{L^-=(l_2<l_3<\dots<l_r)} \psi_{l_2}^M \dots \psi_{l_r}^M \det{}^{\mathrm{col}}(Y_{L^-J^-})\right), \tag{6.24}$$

$\psi_{j_1}^{MY} = \sum_{l_1} \psi_{l_1}^M Y_{l_1 j_1}$, commuting $Y_{l_1 j_1}$ and $\psi_{l_2}^M \dots \psi_{l_r}^M$, we get:

$$= \sum_{l_1} \sum_{L^-=(l_2<l_3<\dots<l_r)} \psi_{l_1}^M \psi_{l_2}^M \dots \psi_{l_r}^M Y_{l_1 j_1} \det{}^{\mathrm{col}}(Y_{L^-J^-}) \tag{6.25}$$

$$+ \sum_{L^-=(l_2<l_3<\dots<l_r)} \left(\sum_{l_1} \psi_{l_1}^M \big[Y_{l_1 j_1}, \psi_{l_2}^M \dots \psi_{l_r}^M\big] + (r-1)\psi_{j_1}^Q \psi_{l_2}^M \dots \psi_{l_r}^M\right) \det{}^{\mathrm{col}}(Y_{L^-J^-}). \tag{6.26}$$

Continuing the chain of equalities we have, thanks to Lemma 16 that the second summands vanishes, and so

$$= \sum_{l_1} \sum_{L^-=(l_2<l_3<\dots<l_r)} \psi_{l_1}^M \psi_{l_2}^M \dots \psi_{l_r}^M Y_{l_1 j_1} \det{}^{\mathrm{col}}(Y_{L^-J^-}). \tag{6.27}$$

[13] Proposition 14 is a toy model of the theorem above; in its simplicity, it is actually a good illustration for the proof of the theorem.

By formula (6.14) (i.e. column expansion of the determinant) we have that this equals

$$= \sum_{L=(l_1<l_2<l_3<\cdots<l_r)} \psi_{l_1}^M \psi_{l_2}^M \ldots \psi_{l_r}^M \det{}^{\mathrm{col}}(Y_{LJ}), \tag{6.28}$$

and, by (6.12), we arrive at

$$= \sum_{L=(l_1<l_2<l_3<\cdots<l_r)} \sum_{I=(i_1<i_2<i_3<\cdots<i_r)} \psi_{i_1} \ldots \psi_{i_r} \det{}^{\mathrm{col}}(M_{IL}) \det{}^{\mathrm{col}}(Y_{LJ}). \tag{6.29}$$

So we transformed the left-hand side (6.20) to the right-hand side of (6.20). Equality (6.20) is equivalent to desired formula (6.8) in the theorem. Hence the theorem is proved. □

Question 1. Consider two Manin matrices M, Y, such that they satisfy Conditions 1, 2 above. Can one develop some linear algebra (Cramer rule, Cayley–Hamilton theorem, etc.) for MY (or $MY+Q$)? We will see below that if Q is zero then MY is a Manin matrix, so the answer is affirmative. Also note that for the Capelli case (i.e. $M_{ij}=x_{ij}$, $Y_{ij}=\partial_{ji}$) it is also true.

6.2. No correction case and new Manin matrices

The Conditions 1 and 2 given above are easy to check for concrete pairs M, Y; however it is not so clear how to parameterize all the solutions in a simple way. Let briefly discuss the simplest case.

Theorem 6 has the following corollary:

Corollary 5. *Let M be a Manin matrix, and Y such that the following holds*:

$$\forall p, j: \quad \sum_{l=1,\ldots,m} \psi_l^M \left[Y_{lj}, \psi_p^M\right] = 0, \tag{6.30}$$

then:

$$\det{}^{\mathrm{col}}(MY) = \det{}^{\mathrm{col}}(M)\det{}^{\mathrm{col}}(Y), \tag{6.31}$$

$$\det{}^{\mathrm{col}}\big((MY)_{IJ}\big) = \sum_{L=(l_1<l_2<l_3<\cdots<l_r,\, l_r\leqslant m)} \det{}^{\mathrm{col}}(M_{IL})\det{}^{\mathrm{col}}(Y_{LJ}). \tag{6.32}$$

Moreover if in addition Y is also a Manin matrix, then MY is a Manin matrix.

Proof. The requirement (6.30) is the simply Condition 1 with $\psi_i^Q=0$. In this instance, the anticommutativity Condition 2, (6.6) holds true in a trivial way. We can apply our Theorem with the matrix Q equal to zero and obtain (6.31) and (6.32).

To prove that MY is a Manin matrix, it is enough to prove that ψ_j^{MY} anticommute. Indeed this is guaranteed by Manin's property (Proposition 1). By (6.12) we have

$$\psi_{j_1}^{MY}\psi_{j_2}^{MY} = \sum_{i_1,i_2} \psi_{i_1}\psi_{i_2} \det{}^{\mathrm{col}}\big((MY)_{(i_1 i_2)(j_1 j_2)}\big) \tag{6.33}$$

$$\text{by (6.32)} = \sum_{i_1,i_2}\sum_{l_1,l_2} \psi_{i_1}\psi_{i_2} \det{}^{\mathrm{col}}\big((M)_{(i_1 i_2)(l_1 l_2)}\big)\det{}^{\mathrm{col}}\big((Y)_{(l_1 l_2)(j_1 j_2)}\big). \tag{6.34}$$

Y is a Manin matrix, so the determinant changes the sign after interchange of columns, and so 6.33 equals

$$-\sum_{i_1,i_2}\sum_{l_1,l_2}\psi_{i_1}\psi_{i_2}\det{}^{\mathrm{col}}\big((M)_{(i_1i_2)(l_1l_2)}\big)\det{}^{\mathrm{col}}\big((Y)_{(l_1l_2)(j_2j_1)}\big). \tag{6.35}$$

Making the transformations in the reverse order we come to:

$$= -\psi^{MY}_{j_2}\psi^{MY}_{j_1}. \tag{6.36}$$

So $\psi^{MY}_{j_2}$ anticommute and thus MY is a Manin matrix. □

Let us now reformulate condition (6.30) in different ways.
It is easy to see that (6.30) is equivalent to:

$$\sum_{l=1,\dots,m} M_{al}[Y_{lj}, M_{bp}] - M_{bl}[Y_{lj}, M_{ap}] = 0. \tag{6.37}$$

Lemma 17. *Assume that we can find such matrices A^{pj}_{**}, that*:

$$\forall l, p, j:\quad \big[Y_{lj}, \psi^M_p\big] = \sum_v \psi^M_v A^{pj}_{vl}, \tag{6.38}$$

then it is straightforward to see that condition (6.30) *is equivalent to*:

$$\forall p, j:\quad A^{pj}_{vl} = A^{pj}_{lv}, \tag{6.39}$$

i.e. $\forall p, j$ matrix A^{pj}_{**} is symmetric.

This is a quite transparent condition, provided one is able to identify the matrices A^{pj}.
The simplest case is the following:

Lemma 18. *Assume that we can find elements f_{lpj}, such that*

$$\forall l, p, j:\quad \big[Y_{lj}, \psi^M_p\big] = \psi^M_l f_{lpj}, \tag{6.40}$$

then condition (6.30) *is obviously satisfied.*

This case corresponds to the previous with A^{pj} being diagonal matrices.

Example 7. Consider $\mathbb{C}[x_{ij}]$, the matrix M: $M_{ij} = x_{ij}$, and the operators $R_{lp} = \sum_k x_{kl}\partial_{kp}$. One can easily see that $[R_{lj}, \psi^M_p] = \delta_{pj}\psi^M_l$. Consider the matrices M, Y:

$$M_{ij} = x_{ij},\ Y_{lj} = \sum_p f_{ljp}(x_{ij})R_{lp}, \tag{6.41}$$

then one can see that they satisfy the condition (6.40) above. So $\det^{\mathrm{col}}(MY) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y)$ and the more general Cauchy–Binet formulae hold true.

6.3. The "Capelli–Caracciolo–Sportiello–Sokal" case

We shall herewith first recall the Capelli identity [11], then its remarkable generalization S. Caracciolo, A. Sportiello, A. Sokal [12] and explain how it can be naturally derived within our formalism.

Consider the polynomial algebra $\mathbb{C}[x_{ij}]$, and the matrices:

$$M = \begin{pmatrix} x_{11} & \dots & x_{1m} \\ \dots & \dots & \dots \\ x_{n1} & \dots & x_{nm} \end{pmatrix}, \qquad Y = \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \dots & \frac{\partial}{\partial x_{s1}} \\ \dots & \dots & \dots \\ \frac{\partial}{\partial x_{1m}} & \dots & \frac{\partial}{\partial x_{sm}} \end{pmatrix}. \tag{6.42}$$

Theorem 7. *(See A. Capelli [11].) If $n = m = s$, then*

$$\det^{\mathrm{col}}\bigl(MY + \mathrm{diag}(n-1, n-2, \dots, 1, 0)\bigr) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y). \tag{6.43}$$

And more generally $(n, m, s$ are arbitrary$)$ the following Cauchy–Binet formulae hold true.

Let $I = (i_1 < i_2 < \dots < i_r)$, $J = (j_1, \dots, j_r)$, be two multi-indexes $i_a \leqslant n$, $j_a \leqslant s$, $r \leqslant n, s$. Then

$$\det^{\mathrm{col}}\bigl((MY)_{IJ} + \mathrm{diag}(r-1, r-2, \dots, 1, 0)\bigr) = \sum_{L=(l_1<l_2<\dots<l_r)} \det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y_{LJ}). \tag{6.44}$$

Recently the following unexpected and general result which includes the Capelli identity as a particular case has been obtained. It concerns matrices satisfying certain commutation condition:

Definition 10. Let us say that two matrices M, Y of size $n \times m$ and $m \times s$ respectively satisfy the Caracciolo–Sportiello–Sokal condition (CSS-condition for brevity), if the following is true:

$$[M_{ij}, Y_{kl}] = -\delta_{jk} Q_{il}, \tag{6.45}$$

for some elements Q_{il}.

In words: elements in jth column of M commute with elements in kth row of Y unless $j = k$, and in this case commutator of the elements M_{ik} and Y_{kl} depends only on i, l, but does not depend on k. (See [12] formula (1.14), noticing that our matrix Y is the *transpose* to their matrix B).

Theorem 8. *(See [12, Proposition 1.2′, page 4].) Let M be an $n \times m$ Manin matrix, and Y be an arbitrary $($not necessarily Manin$)$ $m \times s$ matrix that satisfy the CSS-condition* (10)*. Then $n = m = s$,*

$$\det^{\mathrm{col}}\bigl(MY + Q\ \mathrm{diag}(n-1, n-2, \dots, 1, 0)\bigr) = \det^{\mathrm{col}}(M)\det^{\mathrm{col}}(Y), \tag{6.46}$$

where matrix Q matrix with elements Q_{ij}.

More generally for arbitrary n, m, s the following Cauchy–Binet formulae holds true. Let $I = (i_1 < i_2 < \dots < i_r)$, $J = (j_1, \dots, j_r)$, be two multi-indexes $i_a \leqslant n$, $j_a \leqslant s$, $r \leqslant n, s$. Then:

$$\det^{\mathrm{col}}\bigl((MY)_{IJ} + Q_{IJ}\mathrm{diag}(r-1, r-2, \dots, 1, 0)\bigr) = \sum_{L=(l_1<l_2<\dots<l_r)} \det^{\mathrm{col}}(M_{IL})\det^{\mathrm{col}}(Y_{LJ}). \tag{6.47}$$

Proof. Let us show that CSS-theorem naturally arises from our Theorem 6. To do this we need to check that Conditions 1 and 2 are verified. To this end, let us recall the notations: ψ_i are Grassmann variables (i.e. $\psi_i^2 = 0, \psi_i\psi_j = -\psi_j\psi_i$); ψ_i commute with M_{ij} and Y_{kl}. By ψ_i^M we denote $\sum_k \psi_k M_{ki}$ (see Notation 2).

Condition 1, page 277 reads:

$$\forall p, j\, \exists \psi_j^Q \in \mathcal{K} \otimes \Lambda[\psi_1, \dots, \psi_n]: \quad \sum_{l=1,\dots,m} \psi_l^M \big[Y_{lj}, \psi_p^M\big] = \psi_p^M \psi_j^Q. \tag{6.48}$$

Now we can observe that simplest and most natural way to obtain that $\sum_l \psi_l^M A_l$ (for some $A_l \in \mathcal{K}$) be proportional to ψ_p^M is clearly to require that $A_l = \delta_{lp} B_l$. So let us require that there exists ψ_j^Q such that:

$$\big[Y_{lj}, \psi_p^M\big] = \delta_{lp} \psi_j^Q. \tag{6.49}$$

Lemma 19. *Condition* (6.49) *is exactly equivalent to the CSS-condition* (6.45)*.*

Thus the CSS-condition implies that Condition 1, (6.4) is satisfied for the matrices M, Y.
Now, an unexpected fact holds true:

Lemma 20. *The CSS-condition automatically implies that Condition* 2 (6.6) *is also satisfied, that is* ψ_i^M *and* ψ_j^Q *anticommute.*

Proof. If $n = 1$, then $0 = \psi_i^M \psi_j^Q = -\psi_j^Q \psi_i^M$, so anticommutativity holds by trivial reasons. Assume $n > 1$, take: $l \neq i$ and use use (6.49): $\psi_i^Q = [Y_{lj}, \psi_l^M]$:

$$\begin{aligned}
\psi_i^M \psi_j^h + \psi_j^h \psi_i^M &= \psi_i^M \big[Y_{lj}, \psi_l^M\big] + \big[Y_{lj}, \psi_l^M\big] \psi_i^M &\quad (6.50)\\
&= \psi_i^M Y_{lj} \psi_l^M - \psi_i^M \psi_l^M Y_{lj} + Y_{lj} \psi_l^M \psi_i^M - \psi_l^M Y_{lj} \psi_i^M &\\
&= \text{use (6.49): } \big[\psi_i^M, Y_{lj}\big] = 0 \text{ when } i \neq l &\quad (6.51)\\
&= Y_{lj} \psi_i^M \psi_l^M - \psi_i^M \psi_l^M Y_{lj} + Y_{lj} \psi_l^M \psi_i^M - \psi_l^M Y_{lj} \psi_i^M. &\quad (6.52)
\end{aligned}$$

By Proposition 1 the ψ_i^M anticommute, so the term $Y_{lj} \psi_i^M \psi_l^M + Y_{lj} \psi_l^M \psi_i^M$ cancel each other, and so we can rewrite (6.50) as

$$\psi_i^M \psi_j^h + \psi_j^h \psi_i^M = -\psi_i^M \psi_l^M Y_{lj} - \psi_l^M Y_{lj} \psi_i^M. \tag{6.53}$$

By Manin's property (Proposition 1) the ψ_i^M's anticommute, so we can transform the last relation into

$$\psi_l^M \psi_i^M Y_{lj} - \psi_l^M Y_{lj} \psi_i^M = \psi_l^M \big[\psi_i^M, Y_{lj}\big] = 0, \tag{6.54}$$

once again thanks to (6.49) with $i \neq l$. The lemma is proved. □

Thus the CSS-condition implies both conditions in our Theorem 6, and hence the CSS theorem can be deduced from Theorem 6 above. □

Example 8. It is easy to see that for arbitrary functions $f_{ij}(x_{11}, \dots, x_{nm})$ the matrices below satisfy the CSS-condition:

$$M=\begin{pmatrix} x_{11} & \dots & x_{1m} \\ \dots & \dots & \dots \\ x_{n1} & \dots & x_{nm} \end{pmatrix}, \qquad Y=\begin{pmatrix} \frac{\partial}{\partial x_{11}}+f_{11}(x_{ij}) & \dots & \frac{\partial}{\partial x_{n1}}+f_{n1}(x_{ij}) \\ \dots & \dots & \dots \\ \frac{\partial}{\partial x_{1m}}+f_{1m}(x_{ij}) & \dots & \frac{\partial}{\partial x_{nm}}+f_{nm}(x_{ij}) \end{pmatrix}, \tag{6.55}$$

and so we can apply the theorem for them.

Remark 27. It is obvious that the Capelli identity is a particular case of the CSS-theorem. The first one has been widely studied and generalized (see R. Howe and T. Umeda [56] for classical reference on the subject, and [12] for quite a complete list of references), however, all the generalizations have been related with the Lie algebras, super algebras, quantum groups. It seems it was widely believed that such an identity is intimately related to these in a sense exceptional structures. CSS-theorem shows that it is not true, it is actually a particular case of the more general statement about noncommutative matrices, which actually has nothing to do with Lie algebras or whatever. Let us also remark that [16], Section 4.3.1, contains a very simple proof of the Capelli identity and actually of its generalization E. Mukhin, V. Tarasov, A. Varchenko [87] based on the Schur complement theorem for Manin matrices.

Remark 28. The CSS conditions can also be reformulated with the help of matrix notations:

$$[M\otimes 1, 1\otimes Y]=-(Q\otimes 1)P=-P(1\otimes Q), \tag{6.56}$$

where P is the permutation operator: $P(a\otimes b)=b\otimes a$. However, here we will not explicitly use this property. The Matrix (or Leningrad) notations are discussed in Section 8.

6.4. *Turnbull–Caracciolo–Sportiello–Sokal case*

In 1948 Turnbull [111] proved a Capelli-type identity for symmetric matrices, D. Foata and D. Zeilberger [37] gave a combinatorial proof of this identity. S. Caracciolo, A. Sportiello, A. Sokal [12], Proposition 1.4, proposed a generalization of this result as well. Here we will deduce it from our Theorem 6.

Consider the polynomial algebra $\mathbb{C}[x_{ij}]$, and the *symmetric* matrices:

$$M=\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{13} & x_{23} & x_{33} & \dots & x_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{nn} \end{pmatrix}, \qquad Y=\begin{pmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{13}} & \dots & \frac{\partial}{\partial x_{1n}} \\ \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & \frac{\partial}{\partial x_{23}} & \dots & \frac{\partial}{\partial x_{2n}} \\ \frac{\partial}{\partial x_{13}} & \frac{\partial}{\partial x_{23}} & \frac{\partial}{\partial x_{33}} & \dots & \frac{\partial}{\partial x_{3n}} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial}{\partial x_{1n}} & \frac{\partial}{\partial x_{2n}} & \frac{\partial}{\partial x_{3n}} & \dots & \frac{\partial}{\partial x_{nn}} \end{pmatrix}. \tag{6.57}$$

Theorem 9. *(See H.W. Turnbull [111].)*

$$\det{}^{\mathrm{col}}\big(MY+\mathrm{diag}(n-1,n-2,\dots,1,0)\big)=\det{}^{\mathrm{col}}(M)\det{}^{\mathrm{col}}(Y). \tag{6.58}$$

Definition 11. Let us say that matrices M and Y satisfy the Turnbull–Caracciolo–Sportiello–Sokal condition (TCSS-condition for brevity), if the following is true:

$$[M_{ij}, Y_{kl}]=-h(\delta_{jk}\delta_{il}+\delta_{ik}\delta_{jl}), \tag{6.59}$$

for some element h.

This means that the elements M_{ij} commute with all elements of Y except for elements Y_{ik} and Y_{ki}, and this non-zero commutator is equal to $-h$, for some element h. (See [12], formula (1.25); our M is their A and our Y is their B).

Lemma 21. *Assume $n > 1$. Assume that M is a symmetric $n \times n$ matrix with commuting entries (i.e. $\forall i, j, k, l$: $[M_{ij}, M_{kl}] = 0$), M and Y satisfy the TCSS-condition above. Then*:

$$\forall i, j: \quad [M_{ij}, h] = 0. \tag{6.60}$$

Indeed, let us consider M_{ij}, and pick M_{ab} such that $(ij) \neq (ab)$ and $(ij) \neq (ba)$ (this is possible since $n > 1$). Then $h = [Y_{ab}, M_{ab}]$, by TCSS-condition $[M_{ij}, Y_{ab}] = 0$ and by assumption $[M_{ij}, M_{ab}] = 0$, so $[M_{ij}, h] = [M_{ij}, [Y_{ab}, M_{ab}]] = 0$.

Theorem 10. *(See [12, Proposition 1.4].) Assume M is an $n \times n$* symmetric *matrix with commuting entries,*[14] *Y is $n \times p$ matrix (not necessarily Manin), and matrices M and Y satisfy TCSS-condition. Then*:

If $n = p$,

$$\det{}^{\mathrm{col}}\bigl(MY + h\,\mathrm{diag}(n-1, n-2, \ldots, 1, 0)\bigr) = \det{}^{\mathrm{col}}(M)\det{}^{\mathrm{col}}(Y). \tag{6.61}$$

For arbitrary n, o, more generally the following Cauchy–Binet formulae holds true. Let $I = (i_1 < i_2 < \cdots < i_r)$, $J = (j_1, \ldots, j_r)$, be two multi-indexes $i_a \leqslant n$, $j_a \leqslant n$, $r \leqslant n, p$. Then:

$$\det{}^{\mathrm{col}}\bigl((MY)_{IJ} + h\,\mathrm{diag}(r-1, r-2, \ldots, 1, 0)\bigr) = \sum_{L=(l_1<l_2<\cdots<l_r)} \det{}^{\mathrm{col}}(M_{IL})\det{}^{\mathrm{col}}(Y_{LJ}). \tag{6.62}$$

Proof. We will deduce the theorem above from our Theorem 6. To do this we need to check Conditions 1 and 2. Let us recall the notations: ψ_i are Grassmann variables (i.e. $\psi_i^2 = 0$, $\psi_i\psi_j = -\psi_j\psi_i$) and ψ_i commute with M_{ij} and Y_{kl}. By ψ_i^M we denote $\sum_k \psi_k M_{ki}$ (see Notation 2).

From the TCSS-condition above we see that:

$$\bigl[Y_{lj}, \psi_p^M\bigr] = \delta_{lp} h \psi_j + \delta_{jp} h \psi_l. \tag{6.63}$$

Let us look at Condition 1, (6.4):

$$\sum_{l=1,\ldots,n} \psi_l^M \bigl[Y_{lj}, \psi_p^M\bigr] = \sum_{l=1,\ldots,n} \psi_l^M (\delta_{lp} h \psi_j + \delta_{jp} h \psi_l) \tag{6.64}$$

We preliminary observe that, for any symmetric matrix M (not necessarily a Manin matrix):

$$\sum_{l=1,\ldots,n} \psi_l \psi_l^M = 0. \tag{6.65}$$

Indeed, $\sum_{l=1,\ldots,n} \psi_l \psi_l^M = \sum_{l=1,\ldots,n} \sum_{j=1,\ldots,n} \psi_l \psi_j M_{jl} = \sum_{l<j} \psi_j \psi_l (M_{jl} - M_{lj}) = 0$.

In our case ψ_i and ψ_j^M anticommute so the sum $\sum_{l=1,\ldots,n} \psi_l^M \delta_{jp} h \psi_l$ is zero. Then (6.64) becomes

$$= \sum_{l=1,\ldots,n} \psi_l^M \delta_{lp} h \psi_j = \psi_p^M h \psi_j. \tag{6.66}$$

Hence Condition 1, (6.4) is satisfied for $\psi_j^Q = h\psi_j$.

[14] We could require M to be a Manin matrix, but for the field of characteristic not equal to 2, a symmetric Manin matrix is matrix with commuting entries.

Condition 2, (6.6) requires that ψ_p^M, $h\psi_j$. Assume that $n > 1$, then by Lemma 21 above we know that h, commute with M_{ij}, by definition $\psi_p^M = \sum_k \psi_k M_{kp}$, where ψ_i commute with M_{kl}. These implies Condition 2, for $n > 1$.

Hence our Conditions 1, 2 are satisfied. Applying Theorem 6 we obtain the theorem above for $n > 1$. Since for $n = 1$ the theorem above is the tautology $M_{11}Y_{11} = M_{11}Y_{11}$, the theorem is fully proved. □

6.5. *Generalization to permanents*

We have seen that the identity above has a natural formulation and proof in terms of the Grassmann algebra, so we can also look for similar result for the algebra of polynomials, since both algebras play an equal role in the definition of Manin matrices. Here we will briefly discuss some analogs for permanents of the theorems above. Since proofs are absolutely similar we will give only the formulations and some key comments. One can actually consider the case of super-Manin matrices and then both cases of determinants and permanents are the particular cases of it, but we do not want to overload the text going into the super-theory.

6.5.1. *Preliminaries*

Definition 12. Let us recall that the *column permanent* of a square $n \times n$ matrix A is:

$$\mathrm{per}^{\mathrm{col}} A = \sum_{\sigma \in S_n} A_{\sigma(1)1} A_{\sigma(2)2} \dots A_{\sigma(n)n}, \tag{6.67}$$

i.e. in the product $A_{\sigma(1)1} A_{\sigma(2)2} \dots A_{\sigma(n)n}$ we first take elements from the first column, than the second, and so on and so forth. Here S_n is the permutation group of n letters.

Remark that the definition of permanent is absolutely similar to that of the determinant, without the sign factors given by the parity of the permutation.

Let A be an $n \times m$ matrix. Consider multi-indexes $I = (i_1, \dots, i_{r_1})$, $J = (j_1, \dots, j_{r_2})$. Let us recall (Notation 1) that we denote by A_{IJ} the following $r \times r$ matrix:

$$(A_{IJ})_{ab} = A_{i_a j_b}. \tag{6.68}$$

In formulas below it will be necessary to use the following normalized version of the permanents.

Definition 13. Let A be an $n \times m$ matrix over some (not necessarily commutative ring); let $I = (i_1 \leqslant \dots \leqslant i_r)$, $J = (j_1, \dots, j_r)$, $\forall a$: $j_a \leqslant m$ be multi-indexes (we do not require ordering nor $j_a \neq j_b$). Let us call *normalized* column permanent the quantity: $\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{IJ})$ the following:

$$\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{IJ}) = \frac{1}{(2!)^{n_2}(3!)^{n_3}\dots} \mathrm{perm}^{\mathrm{col}}(A_{IJ}), \tag{6.69}$$

where n_p is defined as follows: $n_p = v$, means that some numbers $a_1, \dots a_v$ enter the sequence I with multiplicity exactly p.

For example for $I = (i_1, i_1, i_2, i_2, i_3, i_3, i_3)$ the factor will be $(2!)^2(3!)$. Note that multiplicities in J do not enter the normalization of the permanent.

Example 9.

$$\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{(11)(11)}) = \frac{1}{2!}\mathrm{perm}^{\mathrm{col}}\begin{pmatrix} A_{11} & A_{11} \\ A_{11} & A_{11} \end{pmatrix} = \frac{1}{2}(A_{11}A_{11} + A_{11}A_{11}) = (A_{11})^2, \tag{6.70}$$

$$\text{more generally:}\quad \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{(aa\ldots a)(bb\ldots b)}) = (A_{ab})^r. \tag{6.71}$$

Notation 3. Consider some elements $x_1, \ldots, x_n$ and an $n \times o$ matrix A, for brevity we denote by x_i^A the element $\sum_{k=1,\ldots,n} x_k A_{ki}$, i.e. just the application of the matrix A to row-vector $(x_1, \ldots, x_n)$:

$$\left(x_1^A, \ldots, x_m^A\right) = (x_1, \ldots, x_n)A. \tag{6.72}$$

The lemma below is quite obvious. It is an analogue of Lemma 15 for Grassmann variables and determinants.

Lemma 22. *Consider commuting variables x_i, $i = 1, \ldots, n$ and an $n \times m$ matrix A; multi-index $J = (j_1, j_2, \ldots, j_r)$ is arbitrary (i.e. it is* not *assumed that $j_a \neq j_b$, nor $j_a < j_b$). Assume that x_i commute with A_{kl}. Then:*

$$x_{j_1}^A x_{j_2}^A \ldots x_{j_r}^A = \sum_{L=(l_1 \leqslant l_2 \leqslant l_3 \leqslant \cdots \leqslant l_r \leqslant n)} x_{l_1} x_{l_2} \ldots x_{l_r} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{LJ}). \tag{6.73}$$

Dropping the assumption $[x_i, A_{kl}] = 0$ we can write the equality in the following way:

$$\sum_{I=(i_1,\ldots,i_r:\ 1 \leqslant i_a \leqslant m)} x_{i_1} x_{i_2} \ldots x_{i_r} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_r j_r} = \sum_{L=(l_1 \leqslant l_2 \leqslant l_3 \leqslant \cdots \leqslant l_r \leqslant n)} x_{l_1} x_{l_2} \ldots x_{l_r} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{LJ}). \tag{6.74}$$

It implies the formula for the expansion of the column permanent with respect to the first column:

$$\begin{aligned} &\sum_{l_1=1,\ldots,n} x_{l_1} \sum_{L^-=(l_2 \leqslant l_3 \leqslant \cdots \leqslant l_r \leqslant n)} x_{l_2} \ldots x_{l_r} A_{l_1 j_1} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{L^- J^-}) \\ &\quad = \sum_{L=(l_1 \leqslant l_2 \leqslant l_3 \leqslant \cdots \leqslant l_r \leqslant n)} x_{l_1} x_{l_2} \ldots x_{l_r} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(A_{LJ}), \end{aligned} \tag{6.75}$$

where $J^- = (j_2, j_3, \ldots, j_r)$.

6.5.2. Cauchy–Binet type formulas for permanents

Proposition 13 *(Cauchy–Binet formula for permanents: first case). Consider an $m \times n$ Manin matrix M and an arbitrary $m \times o$ matrix Y, such that $[M_{ij}, Y_{kl}] = 0$. Consider an arbitrary multi-index $J = (j_1, \ldots, j_r)$ and ordered multi-index $I = (i_1 \leqslant l_2 \leqslant \cdots \leqslant i_r)$, $i_a \leqslant n$. Then:*

$$\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}\left(\left(M^t Y\right)_{IJ}\right) = \sum_{L=(l_1 \leqslant l_2 \leqslant \cdots \leqslant l_r), l_a \leqslant m} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}\left(\left(M^t\right)_{IL}\right) \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}\left((Y)_{LJ}\right). \tag{6.76}$$

Proof. It is quite obvious, but let us nevertheless write it in details, to get a grasp on what is going on.

Consider the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, whose generators satisfy $[x_i, M_{kl}] = 0$ and $[x_i, Y_{kl}] = 0$. By (6.74):

$$x^{M^tY}_{j_1} x^{M^tY}_{j_2} \dots x^{M^tY}_{j_r} = \sum_{I=(i_1 \leqslant i_2 \leqslant i_3 \leqslant \dots \leqslant i_r \leqslant n)} x_{i_1} x_{i_2} \dots x_{i_r} \operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\big(\big(M^tY\big)_{IJ}\big), \tag{6.77}$$

on the other hand, $x^{M^tY}_j = \sum_k x^{M^t}_k Y_{kj}$, and $x^{M^t}_j$ commute by Manin's property (Proposition 1), so again by (6.74):

$$x^{M^tY}_{j_1} x^{M^tY}_{j_2} \dots x^{M^tY}_{j_r} = \sum_{L=(l_1 \leqslant l_2 \leqslant l_3 \leqslant \dots \leqslant l_r), l_r \leqslant m} x^{M^t}_{l_1} x^{M^t}_{l_2} \dots x^{M^t}_{l_r} \operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\big((Y)_{LJ}\big) = \tag{6.78}$$

and again by (6.74):

$$= \sum_{L=(l_1 \leqslant l_2 \leqslant l_3 \leqslant \dots \leqslant l_r), l_r \leqslant m} \;\; \sum_{I=(i_1 \leqslant i_2 \leqslant i_3 \leqslant \dots \leqslant i_r \leqslant n)} x_{i_1} x_{i_2} \dots x_{i_r} \operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\big(\big(M^t\big)_{IL}\big) \operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\big((Y)_{LJ}\big). \tag{6.79}$$

Comparing (6.77) and (6.79) we come to the desired proposition. $\square$

Now, let us weaken the condition of commutativity of the matrix elements M_{kl} and Y_{ij}, and assume the following analogs of Conditions 1 and 2 ((6.4), (6.6)) of Section 6.1 considering:

Condition 1′.

$$\forall p, j \quad \exists x^Q_j \in \mathcal{K} \otimes \mathbb{C}[x_1, \dots, x_n]: \quad \sum_{l=1,\dots,m} x^M_l \big[Y_{lj}, x^M_p\big] = x^M_p x^Q_j, \tag{6.80}$$

note that x^Q_j does not depend on p.

As usual, denote by Q an $n \times s$ matrix corresponding to the elements x^Q_i:

$$\big(x^Q_1, \dots, x^Q_s\big) = (x_1, \dots, x_n) Q. \tag{6.81}$$

Condition 2′. x^M_i and x^Q_j commute:

$$\forall i, j: \quad x^M_i x^Q_j = x^Q_j x^M_i. \tag{6.82}$$

Theorem 11. *Assume M is $m \times n$ Manin matrix, Y is $m \times s$ matrix* (*not necessarily Manin*), *and matrices M^t and Y satisfy conditions* (6.80), (6.82) *above. Then the following Cauchy–Binet formulae holds.*

Let $I = (i_1 \leqslant i_2 \leqslant \dots \leqslant i_r)$, $J = (j_1, \dots, j_r)$,[15] *be two multi-indexes $i_a \leqslant n$, $j_a \leqslant s$, $r \leqslant n, s$. Then*:

[15] Conditions $j_a < j_b$, $j_a \neq j_b$ are *not* required.

$$\begin{aligned}&\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\bigl(\bigl(M^tY\bigr)_{IJ}-Q_{IJ}\operatorname{diag}(r-1,r-2,\ldots,1,0)\bigr)\\&\quad=\sum_{L=(l_1\leqslant l_2\leqslant\cdots\leqslant l_r),l_r\leqslant m}\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\bigl(\bigl(M^t\bigr)_{IL}\bigr)\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}(Y_{LJ}).\end{aligned}\tag{6.83}$$

Here A_{IJ} is the matrix such that $(A_{IJ})_{ab}=A_{i_a j_b}$, (see the Notation 1); *Q is the matrix corresponding to the elements x_i^Q by formula* (6.81) *and elements x_i^Q arise in Condition* 1′ *above*; *by* $\operatorname{diag}(a_1,a_2,\ldots)$ *we denote the diagonal matrix with a_i on the diagonal.*

Let us recall Definition 10 that two matrices M,Y of sizes $n\times m$ and $m\times p$ respectively satisfy the CSS-condition if the following is true:

$$[M_{ij},Y_{kl}]=-\delta_{jk}Q_{il},\tag{6.84}$$

for some elements Q_{il}.

Or “in words”: elements in jth column of M commute with elements in kth row of Y unless $j=k$, and in this case commutator of the elements M_{ik} and Y_{kl} depends only on i,l, but does not depend on k. (See [12] formula (1.14), our Y is *transpose* to their B).

Theorem 12. *Assume $n>1$. Assume M is an $m\times n$ Manin matrix, Y is $m\times s$ matrix* (*not necessarily Manin*), *the and matrices M^t and Y satisfy the CSS-condition* (6.84) *above. Then the following Cauchy–Binet formulae hold true.*

Let $I=(i_1\leqslant i_2\leqslant\cdots\leqslant i_r)$, $J=(j_1,\ldots,j_r)$, be two multi-indexes $i_a\leqslant n$, $j_a\leqslant s$, $r\leqslant n,s$. Then:

$$\begin{aligned}&\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\bigl(\bigl(M^tY\bigr)_{IJ}-Q_{IJ}\operatorname{diag}(r-1,r-2,\ldots,1,0)\bigr)\\&\quad=\sum_{L=(l_1\leqslant l_2\leqslant\cdots\leqslant l_r),l_r\leqslant m}\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}\bigl(\bigl(M^t\bigr)_{IL}\bigr)\operatorname{perm}^{\mathrm{col}}_{\mathrm{norm}}(Y_{LJ}),\end{aligned}\tag{6.85}$$

where Q is the matrix with elements Q_{ij}.

Let us also give an analogue of Lemma 23:

Lemma 23. *For $n>1$ the CSS-condition automatically implies x_i^M and x_j^Q commute* (*and hence that Condition* 2′ *is satisfied*).

Remark 29. The theorem above holds true for $n=1$ under the additional requirement $[M_{11},[M_{11},Y_{1i}]]=0$.

Let us modify the TCSS-condition (6.59) for an antisymmetric matrix M. We impose the following condition:

$$[M_{ij},Y_{kl}]=-h(\delta_{jk}\delta_{il}-\delta_{ik}\delta_{jl}),\tag{6.86}$$

for some element h.

Or “in words”: the elements M_{ij} commute with all elements of Y except with the elements Y_{ik} and Y_{ki}, and these non-zero commutators are equal to $\pm h$ respectively, for some element h.

Theorem 13. *Take $n > 2$, and assume M be an $n \times n$* antisymmetric *matrix with commuting entries (i.e. $\forall i, j, k, l$: $[M_{ij}, M_{kl}] = 0$),*[16] *Y is $n \times s$ matrix (not necessarily Manin), and that the matrices M and Y satisfy TCSS-condition for antisymmetric matrices. Then the following Cauchy–Binet formulae hold true.*

Let $I = (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r)$, $J = (j_1, \ldots, j_r)$, be two multi-indexes $i_a \leqslant n$, $j_a \leqslant n$, $r \leqslant n, s$. Then:

$$\begin{aligned} &\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}\big((MY)_{IJ} - h\,\mathrm{diag}(r-1, r-2, \ldots, 1, 0)\big) \\ &\quad = \sum_{L=(l_1 \leqslant l_2 \leqslant \cdots \leqslant l_r)} \mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(M_{IL})\mathrm{perm}^{\mathrm{col}}_{\mathrm{norm}}(Y_{LJ}). \end{aligned} \tag{6.87}$$

An analogue of Lemma 21 is the following:

Lemma 24. *Take $n > 2$, and assume that M be an antisymmetric $n \times n$ matrix with commuting entries (i.e. $\forall i, j, k, l$: $[M_{ij}, M_{kl}] = 0$), and that M and Y satisfy the TCSS-condition above. Then:*

$$\forall i, j: \quad [M_{ij}, h] = 0. \tag{6.88}$$

Remark 30. The theorem above holds true for $n = 2$ also under the additional requirement $[M_{12}, h] = 0$.

An analogue of the relation (6.65) is the following:

Lemma 25. *For any antisymmetric matrix M (not necessarily Manin matrix):*

$$\sum_{l=1,\ldots,n} x_l x_l^M = 0. \tag{6.89}$$

6.5.3. A Toy model

Let us provide a toy model for the identities above, which is actually a particular case of Theorem 12 (in the case $n = 1$). The proof is extremely simple and it is actually a good illustration of the proof of the main Theorem 6.

Proposition 14. *Assume that some elements M and Y of the ring $\mathcal{K}$ satisfy the following condition: $[M, [M, Y]] = 0$, and denote by $Q = [M, Y]$. Then for any r:*

$$\big(MY - (r-1)Q\big)\big(MY - (r-2)Q\big)\ldots(MY - Q)(MY) = (M)^r(Y)^r. \tag{6.90}$$

Proof. For $r = 1$ it is a tautology. Consider the left-hand side of the equality and by induction:

$$\big(MY - (r-1)Q\big)\big(MY - (r-2)Q\big)\ldots(MY - Q)(MY) = \big(MY - (r-1)Q\big)\big(M^{r-1}Y^{r-1}\big) \tag{6.91}$$

(the formula above is analogous to (6.23) in the proof of the main theorem)

$$= MM^{r-1}YY^{r-1} + M\big[Y, M^{r-1}\big]Y^{r-1} - (r-1)Q\,\big(M^{r-1}Y^{r-1}\big) \tag{6.92}$$

(the formula above is analogous to (6.25) in the proof of the main theorem).

[16] We can require M to be a Manin matrix, but for the field of characteristic not equal to 2, antisymmetric Manin matrix is matrix with commuting entries.

Since $[M, Q] = 0$:

$$\left[Y, M^{r-1}\right] = (r-1)\, Q\, M^{r-1} \tag{6.93}$$

(the formula above is analogous to Lemma 16 in the proof of the main theorem).

Continuing the chain of equalities (6.92) we have:

$$= M M^{r-1} Y Y^{r-1} + M(r-1) Q\, M^{r-1} Y^{r-1} - (r-1) Q \left(M^{r-1} Y^{r-1}\right) = M M^{r-1} Y Y^{r-1}. \tag{6.94}$$

It is right-hand side of the desired equality. The proposition is proved. □

Example 10.

$$\big(\partial_z z - (r-1)\big)\big(\partial_z z - (r-2)\big) \dots (\partial_z z - 1)(\partial_z z) = \partial_z^r z^r. \tag{6.95}$$

Remark 31. Capelli identities related to permanents can be also found in S. Caracciolo, A. Sportiello, A. Sokal [12], Proposition 1.5 (due to Turnbull), M. Nazarov [88], and A. Okounkov [93].[17] Our result is clearly different from the first mentioned result while the relations with the others are not clear to us at the moment.

7. Further properties

In this section we discuss other properties of commutative matrices which can be extended to the case of Manin matrices. Some of them are new like the multiplicativity property of the determinant, the relation of the determinant and the Gauss decomposition, conjugation to the second normal (also called Frobenius, e.g. Wilkinson [65,116]) form. Other properties can be already found, somewhat scattered, in the literature. We include them in order to provide a complete list of properties established at the moment so far Manin matrices and to add some details, comments or different proofs of these results.

7.1. Cayley–Hamilton theorem and the second normal (Frobenius) form

The Cayley–Hamilton theorem can be considered one of the basic results in linear algebra. It was generalized in [16, Theorem 3] to the case of Manin matrices. Let us recall it and present some corollary about conjugation to the second normal (Frobenius) form. Some bibliographic notes are in Section 7.5.

Theorem 14. *Let M be an $n \times n$ Manin matrix. Denote by σ_i, $i = 0, \dots, n$ the coefficients of its characteristic polynomial*: $\det^{\mathrm{col}}(t - M) = \sum_{i=0,\dots,n} (-1)^i \sigma_i t^{n-i}$. *Then*:

$$\sum_{i=0,\dots,n} (-1)^i \sigma_i M^{n-i} = 0, \quad \textit{i.e. } \det^{\mathrm{col}}(t - M)\big|_{t=M}^{\text{right substitute}} = 0. \tag{7.1}$$

If M^t is a Manin matrix, then one can obtain a similar result, using left *substitution and the row determinant*: $\det^{\mathrm{row}}(t - M)\big|_{t=M}^{\text{left substitute}} = 0$.

[17] A. Okounkov considers immanants, i.e. $\sum_{\sigma \in S_n} \chi(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \dots A_{\sigma(n)n}$, where χ is some character of the symmetric group. Permanent and determinant are particular cases of the immanant for $\chi = 1$ and $\chi = (-1)^{sgn(\sigma)}$, respectively.

Remark 32. In the commutative case σ_i is the ith elementary symmetric function of the eigenvalues ($\sigma_i = \sum_{1\leqslant j_1<\cdots<j_i\leqslant n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_i}$). In general: $\sigma_1 = Tr(M)$, $\sigma_n = \det^{\mathrm{col}}(M)$, $\sigma_k = Tr\,\Lambda^k M$.

$$\begin{aligned}\det{}^{\mathrm{col}}(t-M) &= \sum_{i=0,\ldots,n} (-1)^i\sigma_i t^{n-i}\\ &= t^n + t^{n-1}(-1)\sigma_1 + t^{n-2}(+1)\sigma_2 + \cdots + t(-1)^{n-1}\sigma_{n-1} + (-1)^n\sigma_n. \qquad (7.2)\end{aligned}$$

Proof. Proposition 6 shows that $t - M$ admits a classical (left) adjoint matrix $(M - t\ 1_{n\times n})^{adj}$, such that

$$(M-t\ 1_{n\times n})^{adj}(M-t\ 1_{n\times n}) = \det{}^{\mathrm{col}}(M-t\ 1_{n\times n})1_{n\times n}, \qquad (7.3)$$

where, as usual, we denote by $1_{n\times n}$ the identity matrix of size n. The standard idea of proof is very simple: we want to substitute M where t stands; the left-hand side of this equality vanishes manifestly, hence we obtain the desired equality $\det^{\mathrm{col}}(M - t\ 1_{n\times n})|_{t=M} = 0$. The only issue we need to clarify is *how* to substitute M into the equation and why the substitution preserves the equality.

Let us denote by $Adj_k(M)$ the matrices defined by: $\sum_{k=0,\ldots,n-1} Adj_k(M)t^k = (M - t\ 1_{n\times n})^{adj}$. The equality above is an equality of polynomials in the variable t:

$$\begin{aligned}\left(\sum_k Adj_k(M)t^k\right)(M-t1_{n\times n}) &= \sum_k Adj_k(M)Mt^k - \sum_k Adj_k(M)t^{k+1}\\ &= \det{}^{\mathrm{col}}(M-t\ 1_{n\times n}) = (-1)^n \sum_{i=0,\ldots,n}(-1)^i\sigma_i t^{n-i}. \qquad (7.4)\end{aligned}$$

This means that the coefficients of t^i of both sides of the relation coincide. Hence we can substitute $t = M$ in the equality, substituting "from the right":

$$\sum_k Adj_k M M^k - \sum_k Adj_k M^{k+1} = (-1)^n \sum_{i=0,\ldots,n}(-1)^i\sigma_i t^{n-i}\Big|_{t=M}. \qquad (7.5)$$

The left-hand side is manifestly zero, so we obtain the desired equality: $\sum_{i=0,\ldots,n}(-1)^i\sigma_i M^{n-i} = 0$. □

Let us present a corollary on the conjugation of a Manin matrix to the so-called Frobenius normal form.

Corollary 6. *Let M be a $n\times n$ Manin matrix with elements in an associative ring $\mathcal{K}$, and let σ_i, $i = 0,\ldots,n$ be the coefficients of its characteristic polynomial, that is,* $\det^{\mathrm{col}}(t - M) = \sum_{i=0,\ldots,n}(-1)^i\sigma_i t^{n-i}$.

Let $v = (v_1,\ldots,v_n)$ be a vector with elements in $\mathcal{K}$ such that $\forall\, k, l$, v_k commutes with σ_l, (this happens, for instance, if $v_k \in \mathbb{C}$). Let, finally,

$$D = \begin{pmatrix} v\\ vM\\ vM^2\\ \cdots\\ vM^{n-1}\end{pmatrix} = \begin{pmatrix} v_1 & \cdots & v_n\\ \sum_i v_iM_{i1} & \cdots & \sum_i v_iM_{in}\\ \sum_{i,j} v_iM_{ij}M_{j1} & \cdots & \sum_{i,j} v_iM_{ij}M_{jn}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{pmatrix}. \qquad (7.6)$$

Then it holds

$$DM = M_{Frob} D, \quad \textit{where}$$
$$M_{Frob} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ (-1)^{n+1}\sigma_n & (-1)^n \sigma_{n-1} & (-1)^{n-1}\sigma_{n-2} & \dots & -\sigma_2 & \sigma_1 \end{pmatrix}. \tag{7.7}$$

Remark 33. In words, this corollary says that, under the commutativity conditions $[v_k, \sigma_j] = 0$, a Manin matrix M can be conjugated to its normal Frobenius form. Before giving the proof of the corollary let us remark the following. It is easy to see that an arbitrary matrix over a noncommutative ring $\mathcal{K}$ which is which can be embed in a noncommutative field can be conjugated in the form above. (Indeed, this is equivalent to the fact that $n+1$ vectors in a n-dimensional vector space are linearly dependent over an arbitrary field (no need of commutativity) and applying this fact to the vectors $v, Mv, M^2, \dots, M^n v$ one gets the claim. The coefficients of linear dependence precisely appear in the last row of the matrix M_{Frob}.) However, in general the coefficients of the linear dependence will depend on the vector v and they are rational (not polynomial) functions of the matrix entries (that is, they belong to the field of fractions of $\mathcal{K}$, but not to the original ring.) We notice that for Manin matrices, the theorem holds in the same form as in the case of ordinary matrices.

Proof. By definition, the ith ($i = 1, \dots, n$) row DM equals vM^i. The same is true for the first $n-1$ rows of$M_{Frob} D$ equals to vM^i. One only needs to check the equality of the nth row. It equals to vM^n for DM and the Cayley–Hamilton theorem, together with the condition $[v_k, \sigma_l] = 0$ precisely provide that the same expression appears in $M_{Frob} D$. Indeed the nth row of $M_{Frob} D$ equals $\sum_{l=0,\dots,n-1} (-1)^{n-l+1} \sigma_{n-l} v M^l$; thanks to the commutativity condition $[v_k, \sigma_l] = 0$ we can rewrite it as $v(\sum_{l=0,\dots,n-1} (-1)^{n-l+1} \sigma_{n-l} M^l)$. By the Cayley–Hamilton theorem this equals to (vM^n). The corollary is proved. □

Example 11. Consider the 2×2 case, with $v = (0, 1)$. Denote $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$D = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}, \qquad M_{Frob} = \begin{pmatrix} 0 & 1 \\ -(ad - cb) & a + d \end{pmatrix}, \tag{7.8}$$

$$DM = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ ca + dc & cb + d^2 \end{pmatrix}, \tag{7.9}$$

$$M_{Frob} D = \begin{pmatrix} 0 & 1 \\ -(ad - cb) & a + d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ ac + dc & -(ad - cb) + ad + d^2 \end{pmatrix} \tag{7.10}$$

$$= \begin{pmatrix} c & d \\ ac + dc & cb + d^2 \end{pmatrix} = (\text{use: } ac = ca) = \begin{pmatrix} c & d \\ ca + dc & cb + d^2 \end{pmatrix}. \tag{7.11}$$

7.2. Newton and MacMahon–Wronski identities

The aim of this section is to generalize the Newton and MacMahon–Wronski identities to the case of Manin matrices. As we shall see, they hold true exactly in the same form as in the commutative case, which we herewith recall.

There are three basic families of symmetric functions in n variables:

1. $\sigma_k = \sum_{1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n} \prod_{p=1,\dots,n} \lambda_{i_p}$, $i = 1, \dots, n$; they are called the *elementary* symmetric functions.

2. $S_k = \sum_{0 \leqslant i_1,\dots,i_n:\ i_1+\cdots+i_n=k} \prod_{p=1,\dots,n} \lambda_p^{i_p}$, $k > 0$ – the so-called *complete*.
3. $T_k = \sum_{p=1,\dots,n} \lambda_p^k$, $k > 0$ – the so-called power sums.

They can be rewritten in the matrix language as follows:

$$\sigma_k = Tr\, \Lambda^k M \quad \Bigl(\sum_{k=0,\dots,n} (-t)^k \sigma_k = \det{}^{\mathrm{col}}(1 - tM)\Bigr), \tag{7.12}$$

$$S_k = Tr\, S^k M, \tag{7.13}$$

$$T_k = Tr\bigl(M^k\bigr). \tag{7.14}$$

Here M is a matrix with entries in $\mathbb{C}$ with eigenvalues $\lambda_1, \dots, \lambda_n$. $S^k M$ is the symmetric tensor power of M while $\Lambda^k M$ is the antisymmetric power.

Theorem. *σ_k, S_k, T_k are related by the following set of identities for all $k > 0$:*

$$\textit{MacMahon–Wronski}: \quad 0 = \sum_{l=0,\dots,k} (-1)^l S_l \sigma_{k-l} = \sum_{l=0,\dots,k} (-1)^l \sigma_{k-l} S_l, \tag{7.15}$$

$$\textit{Newton}: \quad -(-1)^k k \sigma_k = \sum_{i=0,\dots,k-1} (-1)^i \sigma_i T_{k-i}, \tag{7.16}$$

$$\textit{Second Newton}: \quad k S_k = \sum_{i=0,\dots,k-1} T_{k-i} S_i. \tag{7.17}$$

Our main goal is to explain the following:

Claim. *The formulas above hold true when M is a Manin matrix.*

Remark 34. In the case of Manin matrices the order in products in the formulas above is important. The MacMahon–Wronski identity has been first obtained in S. Garoufalidis, T. Le, D. Zeilberger [39], the Newton one in [16]. Here we will collect these results.

The identities above can be easily reformulated in terms of generating functions:

Corollary. *Let M be a Manin matrix. Denote by $\sigma(t), S(t), T(t)$ the following generating functions:*

$$\sigma(t) = \sum_{k=0,\dots,n} (-t)^k \sigma_k = \det{}^{\mathrm{col}}(1 - tM), \tag{7.18}$$

$$S(t) = \sum_{k=0,\dots,\infty} t^k S_k = \sum_{k=0,\dots,\infty} t^k\, Tr\, S^k M, \tag{7.19}$$

$$T(t) = \sum_{k=0,\dots,\infty} t^k T_{k+1} = \sum_{k=0,\dots,\infty} t^k\, Tr\bigl(M^{k+1}\bigr) = Tr\, \frac{M}{1 - tM}. \tag{7.20}$$

Then the relations between σ_l, S_l, T_l can be written as follows:

$$\textit{MacMahon–Wronski}: \quad 1 = \sigma(t) S(t) = S(t) \sigma(t), \tag{7.21}$$

$$\textit{Newton}:\quad -\partial_t\sigma(t)=\sigma(t)T(t), \tag{7.22}$$

$$\textit{Second Newton}:\quad \partial_t S(t)=T(t)S(t). \tag{7.23}$$

Remark 35. In the commutative case the Newton identities can be reformulated in the other forms. This might not be the case for Manin matrices, since, as we shall see in the sequel, for generic Manin matrices we have:

$$\det{}^{\mathrm{col}}\big(e^{M}\big)\neq e^{Tr(M)}\quad\text{and}\quad\big(\det{}^{\mathrm{col}}(1+M)\big)\neq e^{Tr(\ln(1+M))}.$$

7.2.1. Newton identities

Let us recall the result of [16] and give a detailed proof. Bibliographic notes are in Section 7.5.

Theorem 15. *Let M be $n\times n$ Manin matrix. Denote by σ_i, $i=0,\dots,n$ coefficients of its characteristic polynomial*: $\det^{\mathrm{col}}(1-tM)=\sum_{i=0,\dots,n}(-t)^i\sigma_i$, *conventionally, let $\sigma_k=0$, for $k>n$. Then*:

$$-\partial_t\det{}^{\mathrm{col}}(1-tM)=\big(\det{}^{\mathrm{col}}(1-tM)\big)\sum_{k=0,\dots,\infty}t^k\,Tr\big(M^{k+1}\big) \tag{7.24}$$

$$\Leftrightarrow\quad \forall k\geqslant 0: -(-1)^k k\sigma_k=\sum_{i=0,\dots,k-1}(-1)^i\sigma_i\,Tr\big(M^{k-i}\big). \tag{7.25}$$

If M^t is a Manin matrix, then $\partial_t\det^{\mathrm{row}}(1-tM)=(1/t\sum_{k=0,\dots,\infty}t^k\,Tr(M^{k+1}))(\det^{\mathrm{row}}(1-tM))$.

Remark 36. Using the generating functions $\sigma(t)=\det^{\mathrm{col}}(1-tM)$, $T(t)=Tr\,\frac{M}{1-tM}$, one rewrites 7.24 as

$$-\partial_t\sigma(t)=\sigma(t)T(t). \tag{7.26}$$

One can also rewrite it as: $\frac{1}{t}\partial_t\det^{\mathrm{col}}(t-M)=(\det^{\mathrm{col}}(t-M))\sum_{k=0,\dots,\infty}Tr(M/t)^k$.

Remark 37. In the commutative case σ_i is ith elementary symmetric function of the eigenvalues $(\sum_{j_1<\dots<j_i}\prod\lambda_{j_k})$. In general: $\sigma_1=Tr(M)$, $\sigma_n=\det^{\mathrm{col}}(M)$, $\sigma_k=Tr\,\Lambda^k M$.

$$\det{}^{\mathrm{col}}(1-tM)=\sum_{i=0,\dots,n}(-t)^i\sigma_i. \tag{7.27}$$

The formulae are identical in the Manin case, provided one pays attention to the order of terms: $\sigma_i\,Tr\,M^p$ if M is a Manin matrix, while $Tr\,M^p\sigma_i$ if M^t is a Manin matrix.

Proof of Theorem 15. The proof of the theorem is somewhat a standard one. First we need the following simple lemma:

Lemma 26. *Consider an arbitrary matrix M (i.e., not necessarily a Manin matrix), and define its adjoint matrix M^{adj} in the standard way as follows*: $M^{adj}_{kl}=(-1)^{k+l}\det^{\mathrm{col}}(\widehat{M}_{lk})$ *where $\widehat{M}_{lk}$ is the $(n-1)\times(n-1)$ submatrix of M obtained removing the lth row and the kth column. Then*:

$$Tr(t+M)^{adj}=\partial_t\det{}^{\mathrm{col}}(t+M).$$

Proof. The proof is straightforward, but let us nevertheless write it up.

$$\partial_t \det^{\mathrm{col}}(t+M) = \sum_{i=0,\dots,n-1} (n-i)t^{n-1-i} \sum_{\text{principal } i\times i \text{ submatrices } K\subset M} \det^{\mathrm{col}}(K), \tag{7.28}$$

where a principal submatrix is the one formed by the elements obtained on the intersection of the rows and columns labelled by the same set of indices $j_1,\dots,j_i$,

$$\begin{aligned} Tr(t+M)^{adj} &= \sum_{l=1,\dots,n} \det^{\mathrm{col}}(\widehat{t+M})_{ll} \\ &= \sum_{l=1,\dots,n}\sum_{i=0,\dots,n-1} t^{n-1-i} \sum_{\text{principal } i\times i \text{ submatrices } K \text{ of } \widehat{M}_{ll}} \det^{\mathrm{col}}(K). \end{aligned} \tag{7.29}$$

Clearly, any principal submatrix of $\widehat{M}_{ll}$ is a principal submatrix of M, so one has the same terms in both sums. Moreover submatrices of size i appear as coefficients of t^{n-1-i} in both sums.

So we only need to observe that $\det^{\mathrm{col}}(K)$ for an $i\times i$ submatrix K enters with the same multiplicity in both sums. In the first sum the multiplicity is manifestly $n-i$. Let us look at the second sum. The principal submatrix of size i clearly is a submatrix of $(n-i)$ principal submatrices $(\widehat{t+M})_{ll}$, for example submatrix of size 1, say M_{11}, is a submatrix of $(\widehat{t+M})_{22}, (\widehat{t+M})_{33},\dots,(\widehat{t+M})_{nn}$. So we get that the desired coefficient is $n-i$ and the lemma is proved. □

Remark 38. The same arguments can be applied when dealing with the row-determinant, as well, symmetrized determinant, and so on and so forth.

Let us finalize the proof of the theorem. We have the following chain of relations:

$$\begin{aligned} 1/t \sum_{k=0,\dots,\infty} Tr\big((-M/t)^k\big) &= Tr\,\frac{1}{t+M} = \text{ via Cramer's formula} \\ &= Tr\Big(\big(\det^{\mathrm{col}}(t+M)\big)^{-1}(t+M)^{adj}\Big) = \big(\det^{\mathrm{col}}(t+M)\big)^{-1}\,Tr(t+M)^{adj} \\ &= \text{By Lemma 26} = \big(\det^{\mathrm{col}}(t+M)\big)^{-1}\partial_t \det^{\mathrm{col}}(t+M). \end{aligned} \tag{7.30}$$

This identity gives the identities (7.25), i.e.

$$\forall k \geqslant 0\colon\quad -(-1)^k k\sigma_k = \sum_{i=0,\dots,k-1} (-1)^i \sigma_i\,Tr\big(M^{k-i}\big), \tag{7.31}$$

which in turn is equivalent[18] to formula (7.24):

$$-\partial_t \det^{\mathrm{col}}(1-tM) = \big(\det^{\mathrm{col}}(1-tM)\big) \sum_{k=0,\dots,\infty} t^k\,Tr\big(M^{k+1}\big). \tag{7.32}$$

Theorem 15 on the Newton identities is thus proved. □

[18] In the commutative case one can pass from (7.30) to (7.32) as follows. Substitute $M=N^{-1}$ in (7.30) and use that $\det^{\mathrm{col}}(t+N^{-1}) = \det^{\mathrm{col}}(Nt+1)\det^{\mathrm{col}}(N^{-1})$, one gets: $Tr\,\frac{N}{Nt+1} = (\det^{\mathrm{col}}(Nt+1))^{-1}\det^{\mathrm{col}}(N)\det^{\mathrm{col}}(N^{-1})\partial_t\det^{\mathrm{col}}(Nt+1) = (\det^{\mathrm{col}}(Nt+1))^{-1}\partial_t\det^{\mathrm{col}}(Nt+1)$ changing N to $-N$ one gets (7.32). The open question is whether $\det^{\mathrm{col}}(t+N^{-1}) = \det^{\mathrm{col}}(Nt+1)\det^{\mathrm{col}}(N^{-1})$ for Manin matrices as well.

Remark 39. The case in which M^t is a Manin matrix can be treated in a similar way.

Example 12. It is instructive to explicitly perform the computation in the 2×2 case. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where M is a generic matrix (i.e. not necessarily a Manin matrix). We have:

$$Tr\big(M^2\big) - \sigma_1\, Tr(M) + 2\sigma_2 \tag{7.33}$$

$$= Tr\begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix} - (a+d)^2 + 2(ad - cb) \tag{7.34}$$

$$= \big(a^2 + bc + cb + d^2\big) - \big(a^2 + ad + da + d^2\big) + 2(ad - cb) \tag{7.35}$$

$$= (bc) - (da) + (ad - cb) = [a, d] + [b, c], \tag{7.36}$$

and, similarly,

$$Tr\big(M^3\big) - \sigma_1\, Tr\big(M^2\big) + \sigma_2\, Tr(M) = \big([a, d] + [b, c]\big)a + [c, a]b + [b, d]c.$$

Notice that $-3\sigma_3$ does not appear in the last formula since $\sigma_k = 0$, for $k > 2$ for 2×2 matrices. We see that Manin's relations imply that the expressions above are zeroes.

A No-go fact. In the commutative case there is the well-known identity $\det^{\mathrm{col}}(e^N) = e^{Tr\,N}$, which can be readily seen by diagonalization of N. Substituting $N = \log(1 - tM)$ one obtains: $(\det^{\mathrm{col}}(1 - tM)) = e^{Tr(\ln(1-tM))}$. Here we show that these identities do not hold in the case of Manin matrices; actually, they do not hold even in more restrictive case of Cartier–Foata matrices.

Let us remark that in the commutative case the Newton identity

$$-\partial_t \det^{\mathrm{col}}(1 - tM) = \big(\det^{\mathrm{col}}(1 - tM)\big) \sum_{k=1,\dots,\infty} Tr\big(t^{-1}(Mt)^k\big)$$

easily follows from the identities above. Indeed, deriving the identity $(\det^{\mathrm{col}}(1 - tM)) = e^{Tr(\ln(1-tM))}$, with respect to t, one obtains:

$$\partial_t\big(\det^{\mathrm{col}}(1 - tM)\big) = Tr\big(\ln'(1 - tM)\big)e^{Tr(\ln(1-tM))} = Tr\big(\ln'(1 - tM)\big)\big(\det^{\mathrm{col}}(1 - tM)\big). \tag{7.37}$$

Using $Tr(\ln'(1 - tM)) = -Tr(M(1 - tM)^{-1})$, one arrives to

$$\partial_t\big(\det^{\mathrm{col}}(1 - tM)\big) = -Tr\big(M(t - M)^{-1}\big)\big(\det^{\mathrm{col}}(1 - tM)\big),$$

which is the Newton identity from Theorem 15 above.

One may try to argue in the opposite direction, but the crucial point is the commutativity which is absent for Manin matrices, and was used in the first equality of (7.37). So it is not guaranteed that the exponential relations hold true in the general case of a Manin matrix, as indeed it is proven by the next two counterexamples.

Counterexample 1. Consider a 2×2 matrix M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{7.38}$$

and assume it is a Cartier–Foata matrix, that is, elements from different rows commute. Introduce a formal scalar variable ϵ; clearly enough, $1_{2\times 2} + \epsilon M$ is again a Cartier–Foata matrix.

Fact.

$$\det^{\mathrm{col}}\bigl(\exp(1+\epsilon M)\bigr) \neq \exp\bigl(Tr\bigl(\ln(1+\epsilon M)\bigr)\bigr). \tag{7.39}$$

Actually, the equality holds up to order 2, but not at order 3.

Proof. The coefficient of ϵ^3 in the left-hand side is equal to:

$$\frac{1}{6}\bigl(a^3 + bca + abc + bdc + cab + dcb + cbd + d^3 \tag{7.40}$$

$$+ 3acb + 3ad^2 + 3a^2d + 3bcd - 3cab - 3cbd - 3cab - 3dcb\bigr), \tag{7.41}$$

while that on the right-hand side is $\frac{1}{6}(a+d)^3$. In the Cartier–Foata case $[a,d]=0$, so the difference is given by:

$$\frac{1}{6}(bca + abc + bdc + cab + dcb + cbd \tag{7.42}$$

$$+ 3acb + 3bcd - 3cab - 3cbd - 3cab - 3dcb) \tag{7.43}$$

$$= \frac{1}{6}(bac + bcd - abc - bdc) = \frac{1}{6}\bigl([b,a]c + b[c,d]\bigr). \tag{7.44}$$

Since pairs of elements a, b and c, d may be taken to generate free associative algebras and arranged to give a Cartier–Foata matrix, we see that this does not vanish. □

Counterexample 2. Consider M, ϵ as above; then

$$\det^{\mathrm{col}}(1+\epsilon M) \neq e^{Tr(\ln(1+\epsilon M))}. \tag{7.45}$$

The equality holds up to order 2.

The proof of this fact goes exactly as that of Counterexample 1.

7.2.2. MacMahon–Wronski relations

For the sake of completeness let us briefly discuss the so-called MacMahon–Wronski formula for Manin matrices. It was first obtained in [39]. In the language of symmetric functions this identity relates the elementary and the complete symmetric functions. We will provide some more detailed bibliographic notes in Section 7.5.

For an $n \times n$-matrix M over a commutative ring the MacMahon–Wronski identity reads

$$1/\det^{\mathrm{col}}(1-M) = \sum_{k=0,\dots,\infty} Tr\, S^k M, \tag{7.46}$$

where $S^k M$ is kth symmetric power of M. It can be easily verified by diagonalizing the matrix M. As it was mentioned before, $Tr\, S^k M$ is the *complete* symmetric function of the eigenvalues λ_i of M, i.e. $S_k = \sum_{0\leqslant i_1,\dots,i_n:\ i_1+\dots+i_n=k} \prod_{p=1,\dots,n} \lambda_p^{i_p}$.

Theorem. *(See [39].) The MacMahon–Wronski identity holds true for Manin matrices*[19] *provided one defines*

$$Tr\, S^k M = 1/k! \sum_{l_1,\dots l_k:1\leqslant l_i\leqslant n} \mathrm{perm}^{\mathrm{row}} \begin{pmatrix} M_{l_1 l_1} & M_{l_1 l_2} & \dots & M_{l_1 l_k} \\ M_{l_2 l_1} & M_{l_2 l_2} & \dots & M_{l_2 l_k} \\ \dots & \dots & \dots & \dots \\ M_{l_k l_1} & M_{l_k l_2} & \dots & M_{l_k l_k} \end{pmatrix}. \tag{7.47}$$

We remark that the range of summation in this formula allows repeated indexes: $l_{i_1} = l_{i_3} = \cdots$. The permanent of a Manin matrix was defined by formula 3.32 as follows:

$$\mathrm{perm} M = \mathrm{perm}^{\mathrm{row}} M = \sum_{\sigma\in S_n} \prod_{i=1,\dots,n} M_{i\,\sigma(i)}. \tag{7.48}$$

Due to the property that the permanent of a Manin matrix does not change under any permutation of columns, one can rewrite the formula above with the summation without repeated indexes:

$$Tr\, S^k M = 1/k! \sum_{1\leqslant l_1\leqslant\dots\leqslant l_k\leqslant n} n_1!n_2!\dots n_l!\mathrm{perm}^{\mathrm{row}} \begin{pmatrix} M_{l_1 l_1} & M_{l_1 l_2} & \dots & M_{l_1 l_k} \\ M_{l_2 l_1} & M_{l_2 l_2} & \dots & M_{l_2 l_k} \\ \dots & \dots & \dots & \dots \\ M_{l_k l_1} & M_{l_k l_2} & \dots & M_{l_k l_k} \end{pmatrix}, \tag{7.49}$$

where n_i is the set of multiplicities of the set $(l_1,\dots,l_k)$, i.e. n_i is multiplicity of the number i in the set $(l_1,\dots,l_k)$.

One sees that the definition of traces of symmetric powers is the same as in the commutative case, with the proviso in mind to use row permanents.

As an immediate consequence of formula 7.46 we get

Corollary 7. *For all $p > 0$ it holds*

$$0 = \sum_{l=0,\dots,p} (-1)^l S_l \sigma_{p-l} = \sum_{l=0,\dots,p} (-1)^l \sigma_{p-l} S_l, \tag{7.50}$$

where $S_k = Tr\, S^k M$ and σ_k are the coefficients of the characteristic polynomial $\det^{\mathrm{col}}(1 - tM) = \sum_{l=0,\dots,n} (-1)^l t^l \sigma_l$.

Example 13. Let us explicitly write the relation: $S_2 - S_1\sigma_1 + \sigma_2 = 0$, for a 2×2 matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

We have:

$$S_1 = a + d, \qquad \sigma_1 = a + d, \qquad \sigma_2 = ad - cb, \tag{7.51}$$

while

[19] One can find in [39] more general case of q-Manin matrices.

$$S_2 = \frac{1}{2}\left(\operatorname{perm}\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \operatorname{perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \operatorname{perm}\begin{pmatrix} d & c \\ b & a \end{pmatrix} + \operatorname{perm}\begin{pmatrix} d & d \\ d & d \end{pmatrix}\right) \tag{7.52}$$

$$= \frac{1}{2}\left(2a^2 + ad + bc + da + cb + 2d^2\right) \tag{7.53}$$

using only one Manin's relation: $[a,d] = [c,b]$:

$$= \left(a^2 + ad + bc + d^2\right). \tag{7.54}$$

Thus:

$$S_2 - S_1\sigma_1 + \sigma_2 = \left(a^2 + ad + bc + d^2\right) - (a+d)^2 + ad - cb \tag{7.55}$$

$$= \left(a^2 + ad + bc + d^2\right) - \left(a^2 + ad + da + d^2\right) + ad - cb$$

$$= (bc) - (da) + ad - cb = [a,d] + [b,c] = 0. \tag{7.56}$$

7.2.3. Second Newton identities

Theorem 16. *Let M be a Manin matrix. Let $S_k = Tr\, S^k M$ be defined by* (7.47), *$S(t) = \sum_{k=0,\ldots,\infty} t^k S_k$, and $T(t) = \sum_{k=0,\ldots,\infty} t^k\, Tr(M^{k+1}) = Tr\, \frac{M}{1-tM}$. Then the following identities hold*:

$$\partial_t S(t) = T(t)S(t) \quad \Leftrightarrow \quad \forall k > 0\colon \quad kS_k = \sum_{i=0,\ldots,k-1} Tr\left(M^{k-i}\right)S_i.$$

If M^t is a Manin matrix, similar formulas hold with the reverse $\partial_t S(t) = S(t)T(t)$, and the use of column-permanents in formula (7.47).

Proof. The proof follows from the Newton and MacMahon–Wronski identities above. Indeed, with the definition $\sigma(t) = \det^{\text{col}}(1 - tM)$, the MacMahon–Wronski identities read

$$\sigma(t) = S(t)^{-1},$$

while the Newton identities read $-\partial_t\sigma(t) = \sigma(t)T(t)$. Substituting $-\partial_t\sigma(t) = -\partial_t S(t)^{-1} = +S(t)^{-1}(\partial_t S(t))S(t)^{-1}$, we get

$$S(t)^{-1}\left(\partial_t S(t)\right)S(t)^{-1} = S(t)^{-1}T(t),$$

and hence $\partial_t S(t) = T(t)S(t)$. $\square$

7.3. Plücker relations

Here we recall the simplest version of the Plücker identities for Manin matrices (see [76]). They actually follow immediately from the coaction characterization of Manin matrices (Proposition 1, page 249). We will provide some bibliographic notes on various noncommutative Plücker coordinates in Section 7.5.

Proposition 15. *Consider a 2×4 matrix A, assume A^t is a Manin matrix, and let π_{ij} be the minors made from the ith and jth columns* (*minors are calculated as row-determinants, since A^t is a Manin matrix, rather than A*)*. Then*:

$$(\pi_{12}\pi_{34} + \pi_{34}\pi_{12}) - (\pi_{13}\pi_{24} + \pi_{24}\pi_{13}) + (\pi_{14}\pi_{23} + \pi_{23}\pi_{14}) = 0. \tag{7.57}$$

Proof. The proof is the same as in the commutative case. Consider the Grassmann algebra $\mathbb{C}[\psi_1, \ldots, \psi_4]$, and the variables $\tilde{\psi}_1, \tilde{\psi}_2$ defined as:

$$\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = A \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{7.58}$$

It is clear that $\tilde{\psi}_1 \wedge \tilde{\psi}_2 = \sum_{i<j} \pi_{ij} \psi_i \wedge \psi_j$ By Proposition 1 $\tilde{\psi}_1, \tilde{\psi}_2$ are again Grassmann variables, so $(\tilde{\psi}_1 \wedge \tilde{\psi}_2)^2 = 0$. Writing this equation explicitly one arrives at the Plücker relations (7.57). □

7.4. Gauss decomposition and the determinant

Here we show that the determinant of a Manin matrix can be expressed via the diagonal part of the Gauss decomposition exactly in the same way as in the commutative case.

Proposition 16. *Let M be a Manin matrix, assume it can be factorized into Gauss form*:

$$M = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix}. \tag{7.59}$$

Then

$$\det^{\mathrm{col}}(M) = y_n \ldots y_1. \tag{7.60}$$

Analogously for

$$M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x'_{\alpha\beta} & & 1 \end{pmatrix} \begin{pmatrix} y'_1 & & 0 \\ & \ddots & \\ 0 & & y'_n \end{pmatrix} \begin{pmatrix} 1 & & z'_{\beta\alpha} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \tag{7.61}$$

it is true:

$$\det^{\mathrm{col}}(M) = y'_1 \ldots y'_n. \tag{7.62}$$

Example 14.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}, \tag{7.63}$$

$$\det^{\mathrm{col}}(M) = ad - cb = d\bigl(a - bd^{-1}c\bigr), \tag{7.64}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}, \tag{7.65}$$

$$\det^{\mathrm{col}}(M) = ad - cb = a\bigl(d - ca^{-1}b\bigr). \tag{7.66}$$

Proof. According to I. Gelfand, V. Retakh [43, Theorem 2.2.5, page 14] for any noncommutative M it is true that $y_k = |M_{(k)}|_{kk}$, where $M_{(k)}$ is the submatrix of M with $k, k+1, k+2, \dots, n$ rows and columns, and $|N|_{kk}$ is quasideterminant, which is by definition inverse to a corresponding element of N^{-1}. By Cramer's formula (see Proposition 6) $|M_{(k)}|_{kk} = (\det^{\mathrm{col}}(M_{(k)})^{-1}\det^{\mathrm{col}}(M_{(k+1)}))^{-1}$, hence straightforward multiplication and cancellation gives the desired result.

The same arguments for the second equality (see I. Gelfand, S. Gelfand, V. Retakh, R. Wilson [45, Theorem 4.9.7]). $\square$

Remark 40. We are not discussing applications to integrability in this paper. However, let us mention that the result applied to the Yangian generating matrix $e^{-\partial_z}T(z)$ (which is a Manin matrix) gives the useful fact relating the $q\det^{\mathrm{col}}(T(z))$ and the diagonal of the Gauss decomposition.

7.5. Bibliographic notes

The Cayley–Hamilton theorem. The Cayley–Hamilton-like theorems for noncommutative matrices are an object of numerous papers. Enough to say that "Cayley–Hamilton" occurs 179 times during the MathSciNet search. There are also some similar theorems in non-matrix settings. A.J. Bracken and H.S. Green [8,52] found the first examples of related identities for classical semisimple Lie algebras. The subject was further developed in subsequent papers by Australian group (see, e.g. [48, 90]). This type of identities plays an important role in applications (e.g. [60,73]). M. Gould, R. Zhang, A. Bracken [51] (see formula 29, page 2300) extended results to $U_q(g)$ for semisimple g, see also [50]. H. Ewen, O. Ogievetsky, and J. Wess [29, Section 4, Lemma 4.1], contains CH identity for $Fun(GL_{p,q}(2))$. M. Nazarov and V. Tarasov [89] (see also [79, Section 4.3, page 37], [81]) found new approach to Bracken–Green type identities via the Yangian and relation with the Capelli determinant was understood. A. Kirillov [65] (see also [101]) generalized the CH identities related to $U(gl(n))$. T. Umeda [112, Section 3, page 3174], and M. Itoh [59] gave another more direct approach to CH theorem for semisimple Lie algebras and generalizations. I. Kantor, I. Trishin [61] (see also [114]) considered the case of super-matrices. A comprehensive study of Cayley–Hamilton and related identities was attempted in D. Gurevich, A. Isaev, O. Ogievetsky, P. Pyatov, P. Saponov papers [53,97]. A non-trivial character of these identities (in general) is that instead of the matrix power M^k one considers the so-called quantum matrix powers introduced first by J.-M. Maillet [74]. This quantum powers are important from the point of view of quantum integrable systems since their traces provide the commuting elements (integrals of motion), while the traces of usual powers do not commute in general. In all these papers the CH theorem states the linear dependence of power (or quantum powers) of matrices where coefficients of linear dependence are elements from the basic ring. Another version of CH theorem was proposed by J.J. Zhang [118] for the case of quantum group $Fun_q(GL_n)$. In this paper the coefficients of linear dependence are diagonal matrices, with different (in general) elements on the diagonal, but they are not elements of the ground ring like in theorems above.

A quasideterminant version of the CH theorem and the Capelli formula has appeared in [42] (for matrices with coefficients in an arbitrary ring). gl_n-case was detailed in [44, Section 8.6, page 96]. The paper by O. Ogievetsky, A. Vahlas [91] compares the two formulations of the CH theorem (for quantum and for usual powers). The first formulation is more useful from the point of view of integrable systems.

There are also works of more ring theoretic spirit. C. Procesi [96] proves that an appropriate version of CH identity in ring $\mathcal{K}$ with a trace is necessary and sufficient condition for existence of embedding of $\mathcal{K}$ into matrices $Mat(C)$, where C is commutative ring. The paper J. Szigeti [107] discusses another generalization of CH theorem for rings $\mathcal{K}$ such that $\exists n$: $\forall x_1, \dots, x_n$: $[x_1[x_2 \dots [x_{n-1}, x_n] \dots]] = 0$, his result states that there exists a polynomials χ_n such that $\chi_n(A) = 0$ for any matrix A over $\mathcal{K}$. (See also M. Domokos [24] for developments and concrete examples.) In [108] the generalization to the case of matrices with values in $[R, R]$ is discussed.

The Newton identities. The identities for Manin matrices seems to appear first in [16]. For some other classes of matrices with noncommutative entries they can be found in I.M. Gelfand, D. Krob,

A. Lascoux, B. Leclerc, V.S. Retakh, and J.-Y. Thibon [44], D. Gurevich, A. Isaev, O. Ogievetsky, P. Pyatov, P. Saponov papers [53,97], T. Umeda [112], M. Ito [58], see also A. Molev [79], Section 4.1, page 36, and bibliographic notes on page 43, [81], Section 7.1, for quantum matrices see M. Domokos, T.H. Lenagan [25]. The Perelomov–Popov formulas [95] can also seen as a form of the Newton identities. Applying the Newton identities above to the example defined by the first formula of (3.41), page 257, one can probably derive these results.

The MacMahon–Wronski identity. As we have mention the identity for q-Manin matrices has been discovered in S. Garoufalidis, T. Le, D. Zeilberger [39], (see also [28,36,68]). The most natural and simple proof based on Koszul duality has been obtained in Phung Hai, M. Lorenz [54]. For [98] quantum group matrices $RTT = TTR$ and some more general algebras the identity has been already known in some form A. Isaev, O. Ogievetsky, P. Pyatov [57, page 9] (see also [53,97]). For some other noncommutative matrices the identity is discussed in T. Umeda [113], where similar ideas on the Koszul duality are used.

The Plücker coordinates. For generic noncommutative matrices quasi-Plücker coordinates were studied by I. Gelfand, V. Retakh [43, Section 2.1, page 9] (I. Gelfand, S. Gelfand, V. Retakh, R. Wilson [45, Section 4, page 33]). Note that in the commutative case they are not the standard Plücker coordinates, but their ratios. Quasi-Plücker coordinates were further studied in. A. Lauve [70]. It would be interesting to clarify relations between Plücker coordinates above and quasi-Plücker ones.

Due to relations A. Berenstein, A. Zelevinsky [6] for the dual canonical (crystal) basis, the quantum minors and relations between them for [98] quantum group matrices $RTT = TTR$ are widely studied: [10,33,47,64,72,83,103,106]. But these important results are specific to $RTT = TTR$ matrices, one cannot expect that they hold true for more general class of Manin matrices. It might be natural to ask whether any of the properties of q-Plücker coordinates survives in a "twicely" wider class of q-Manin matrices? (Surely not all of them do.)

8. Matrix (Leningrad) form of the defining relations for Manin matrices

In this section we present the defining relations for Manin matrices in the matrix (Leningrad) notations as well as some applications. Such notations are an almost universally used tool in the theory of quantum groups and of quantum integrable system. We shall herewith frame the definition and the main properties of Manin matrices within this formalism. The main benefit is that some formulas (e.g. the commutation relations between the generators) will be most compactly written. Also, we will show how some of our statements can be conveniently translated and used in this formalism.

At first we shall collect some notions coming from the "Leningrad school"'s approach to these issues. Then we shall consider the case of Manin matrices and finally give a few applications.

8.1. A brief account of matrix (Leningrad) notations

The notations are briefly discussed in various texts, let us only mention L. Faddeev, L. Takhtajan [30], V. Chari, A. Pressley [14, Section 7.1C, page 222]. Here we provide definitions and examples.

Let $\mathcal{K}$ be an associative algebra over $\mathbb{C}$. An $n \times n$ matrix $A \in Mat_n[\mathcal{K}]$ with entries A_{ij} in a noncommutative algebra $\mathcal{K}$ can be considered as an element:

$$A = \sum_{ij} A_{ij} \otimes E_{ij} \in \mathcal{K} \otimes Mat_n, \tag{8.1}$$

where E_{ij} are standard "matrix units", i.e. those matrices whose (i, j)th element is 1, and all the others are zero. Mat_n is an associative algebra of $n \times n$ matrices over $\mathbb{C}$.

One considers the tensor product[20] $\mathcal{K} \otimes Mat_n \otimes Mat_n$ and introduces the following notations:

$$\overset{1}{A} = A \otimes 1 = \sum_{ij} A_{ij} \otimes E_{ij} \otimes 1_{n\times n} \in \mathcal{K} \otimes Mat_n \otimes Mat_n, \tag{8.2}$$

$$\overset{2}{B} = 1 \otimes B = \sum_{ij} B_{ij} \otimes 1_{n\times n} \otimes E_{ij} \in \mathcal{K} \otimes Mat_n \otimes Mat_n, \tag{8.3}$$

where $1_{n\times n}$ is identity matrix of size $n \times n$. For further use, we notice that the permutation matrix $P \in Mat_n \otimes Mat_n$, defined by

$$P(e_i \otimes e_j) = e_j \otimes e_i, \tag{8.4}$$

can be written as $P = \sum_{ij} E_{ij} \otimes E_{ji}$, and satisfies the properties:

$$P^2 = 1, \tag{8.5}$$

$$(A \otimes 1)P = P(1 \otimes A), \qquad (1 \otimes A)P = P(A \otimes 1). \tag{8.6}$$

A crucial observation is that, in general $(A \otimes 1)(1 \otimes B) \neq (1 \otimes B)(A \otimes 1)$, and all n^4 commutators between the elements A_{ij} and B_{kl} are encoded in the expression $[A \otimes 1, 1 \otimes B]$. Indeed,

$$[A \otimes 1, 1 \otimes B] = \sum_{ijkl} [A_{ij}, B_{kl}] \otimes E_{ij} \otimes E_{kl} \in \mathcal{K} \otimes Mat_n \otimes Mat_n, \tag{8.7}$$

$$(A \otimes 1)(1 \otimes B) = \sum_{ijkl} A_{ij} B_{kl} \otimes E_{ij} \otimes E_{kl} \in \mathcal{K} \otimes Mat_n \otimes Mat_n, \tag{8.8}$$

$$(1 \otimes B)(A \otimes 1) = \sum_{ijkl} B_{kl} A_{ij} \otimes E_{ij} \otimes E_{kl} \in \mathcal{K} \otimes Mat_n \otimes Mat_n. \tag{8.9}$$

8.1.1. Matrix (Leningrad) notations in 2×2 case

It is useful to exemplify the matrix (Leningrad) notations by 2×2 examples. Although these notions are standard, it might be convenient for the reader to reproduce them here.

$Mat_n \otimes Mat_n$ can be identified with Mat_{n^2}, provided an order of basis elements $e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$ is chosen. As it is customary, we order the basis in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ as $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. Thus

$$A \otimes 1 = \begin{pmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & A_{11} & 0 & A_{12} \\ A_{21} & 0 & A_{22} & 0 \\ 0 & A_{21} & 0 & A_{22} \end{pmatrix}, \qquad 1 \otimes B = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{pmatrix}, \tag{8.10}$$

$$(A \otimes 1)(1 \otimes B) = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}, \tag{8.11}$$

[20] All tensor products are taken over $\mathbb{C}$.

$$(1\otimes B)(A\otimes 1)=\begin{pmatrix} BA_{11} & BA_{12} \\ BA_{21} & BA_{22} \end{pmatrix}=\begin{pmatrix} B_{11}A_{11} & B_{12}A_{11} & B_{11}A_{12} & B_{12}A_{12} \\ B_{21}A_{11} & B_{22}A_{11} & B_{21}A_{12} & B_{22}A_{12} \\ B_{11}A_{21} & B_{12}A_{21} & B_{11}A_{22} & B_{12}A_{22} \\ B_{21}A_{21} & B_{22}A_{21} & B_{21}A_{22} & B_{22}A_{22} \end{pmatrix}. \tag{8.12}$$

The permutation matrix $P=E_{11}\otimes E_{11}+E_{12}\otimes E_{21}+E_{21}\otimes E_{12}+E_{22}\otimes E_{22}$ reads

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.13}$$

8.2. Manin's relations in the matrix form

Here we present a basic lemma which encodes the definition of Manin matrices in matrix notations. All the commutation relations between M_{ij} are encoded in one equation, whose form does not depend on the size n of matrices. The lemma has been suggested to us by P. Pyatov.

Let M be a $n\times n$ matrix with elements in an associative algebra $\mathcal{K}$ over $\mathbb{C}$.

Lemma 27. *A matrix M is a Manin matrix* **iff** *any of the following equivalent formulas hold*:

$$[M\otimes 1, 1\otimes M]=P[M\otimes 1, 1\otimes M], \tag{8.14}$$

$$\frac{(1-P)}{2}(M\otimes 1)(1\otimes M)\frac{(1-P)}{2}=\frac{(1-P)}{2}(M\otimes 1)(1\otimes M), \tag{8.15}$$

$$\frac{(1+P)}{2}(1\otimes M)(M\otimes 1)\frac{(1+P)}{2}=(M\otimes 1)(1\otimes M)\frac{(1+P)}{2}, \tag{8.16}$$

$$(1-P)(1\otimes M)(M\otimes 1)(1+P)=0. \tag{8.17}$$

It should perhaps be noticed that $(1-P)/2$ and $(1+P)/2$ are two orthogonal idempotents, namely the antisymmetrizer and the symmetrizer.

Here we allowed ourselves some abuse of notations, denoting by 1 the identity matrix in Mat_n (e.g. $1\otimes M$) as well as the identity matrix in $Mat_n\otimes Mat_n$ (e.g. $1\pm P$).

Letting $P\in Mat_n\otimes Mat_n$ be the permutation matrix $P(a\otimes b)=b\otimes a$, the formulas above are equalities in the associative algebra $\mathcal{K}\otimes Mat_n\otimes Mat_n$. Namely $M\otimes 1$ is a shorthand notation for $\sum_{i,j} M_{ij}\otimes E_{ij}\otimes 1$ and $1\otimes M$ is $\sum_{i,j} M_{ij}\otimes 1\otimes E_{ij}$ where E_{ij} are standard "matrix units".

Proof. Let us prove (1) $\Leftrightarrow$ (M is a Manin matrix).

$$[M\otimes 1, 1\otimes M]=\sum_{i,j,k,l}[M_{ij}, M_{kl}]\otimes E_{ij}\otimes E_{kl}, \tag{8.18}$$

$$\begin{aligned} P[M\otimes 1, 1\otimes M] &= \Big(\sum_{a,b} E_{ab}\otimes E_{ba}\Big)\Big(\sum_{i,j,k,l}[M_{ij}, M_{kl}]\otimes E_{ij}\otimes E_{kl}\Big) \\ &= \Big(\sum_{a,b}\sum_{i,j,k,l}[M_{ij}, M_{kl}]\otimes E_{ab}E_{ij}\otimes E_{ba}E_{kl}\Big) \end{aligned} \tag{8.19}$$

$$= \left(\sum_{a,b} \sum_{i,j,k,l} [M_{ij}, M_{kl}] \otimes E_{aj}\delta_{bi} \otimes E_{bl}\delta_{ak} \right) \tag{8.20}$$

$$= \left(\sum_{i,j,k,l} [M_{ij}, M_{kl}] \otimes E_{kj} \otimes E_{il} \right) = \left(\sum_{i,j,k,l} [M_{kj}, M_{il}] \otimes E_{ij} \otimes E_{kl} \right). \tag{8.21}$$

Thus $[M \otimes 1, 1 \otimes M] = P[M \otimes 1, 1 \otimes M]$ is equivalent to $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$, which is the definition of Manin matrices, and (1) $\Leftrightarrow$ (M is a Manin matrix) is proved.

To derive that the properties 1,2,3,4 are equivalent to each other is trivial use of the properties: $(M \otimes 1)P = P(1 \otimes M)$, $(1 \otimes M)P = P(M \otimes 1)$. $\square$

Corollary 8. *Let M or M^t be a Manin matrix. Then*:

$$[M \otimes 1, 1 \otimes M]^2 = 0. \tag{8.22}$$

Proof. Let us consider M to be Manin matrix, the other case being similar.

$$[M \otimes 1, 1 \otimes M] = P[M \otimes 1, 1 \otimes M], \quad \text{let us square this equality:} \tag{8.23}$$

$$[M \otimes 1, 1 \otimes M]^2 = P[M \otimes 1, 1 \otimes M]P[M \otimes 1, 1 \otimes M], \tag{8.24}$$

$$P[M \otimes 1, 1 \otimes M]P = [PM \otimes 1P, P1 \otimes MP] = [1 \otimes M, M \otimes 1] = -[M \otimes 1, 1 \otimes M], \tag{8.25}$$

$$\text{so:} \quad [M \otimes 1, 1 \otimes M]^2 = -[M \otimes 1, 1 \otimes M]^2. \qquad \square \tag{8.26}$$

We will show below (see Corollary 29) the following:

Proposition 17. *Let M be a two-sided invertible Manin matrix. Then*

$$0 = P\big[\overset{1}{M}{}^{-1}, \overset{2}{M}{}^{-1}\big][\overset{1}{M}, \overset{2}{M}] = P[\overset{1}{M}, \overset{2}{M}]\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1}[\overset{1}{M}, \overset{2}{M}]. \tag{8.27}$$

It is actually equivalent to the theorem that the inverse to a two-sided invertible Manin matrix is again a Manin matrix.

The following generalization of Lemma 27 has been discovered by A.Silantiev:

Proposition 18. *Let M be an $n \times n$ Manin matrix. Then $\forall m$*:

$$A_m \overset{1}{M} \overset{2}{M} \cdots \overset{m}{M} = A_m \overset{1}{M} \overset{2}{M} \cdots \overset{m}{M} A_m, \tag{8.28}$$

$$\overset{1}{M} \overset{2}{M} \cdots \overset{m}{M} S_m = S_m \overset{1}{M} \overset{2}{M} \cdots \overset{m}{M} S_m, \tag{8.29}$$

where we consider the tensor product $\mathcal{K} \otimes Mat_n^{\otimes m}$, $\overset{k}{M}$ is $\sum_{ij} M_{ij} \otimes 1 \otimes \cdots \otimes E_{ij} \otimes 1 \otimes \cdots \otimes 1$, where E_{ij} stands on kth position. S_m, (A_m) is (anti)-symmetrizer in $(\mathbb{C}^n)^{\otimes m}$, i.e. permutation group S_m naturally acts on $(\mathbb{C}^n)^{\otimes m}$ and A_m is an image of $\sum_{\sigma \in S_m} (-1)^\sigma \sigma$ and S_m is an image of $\sum_{\sigma \in S_m} \sigma$ under this action.

The proof will be provided in [20].

8.3. Matrix (Leningrad) notations in the Poisson case

Recall (see Section 3.1.1 page 248) that by a Poisson–Manin matrix we call a matrix with entries in some Poisson algebra, such that the Manin relations $\{M_{ij}, M_{kl}\} = \{M_{kj}, M_{il}\}$ are satisfied by the *Poisson brackets* between the corresponding elements.

Let us give a Poisson version of Lemma 27:

Lemma 28. *A Matrix M is a Poisson–Manin matrix iff:*

$$\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\} = P\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}, \tag{8.30}$$

while the matrix M^t is a Poisson–Manin matrix iff:

$$\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\} = \{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}P. \tag{8.31}$$

Here we have used the matrix (Leningrad) notation $\{\overset{1}{A} \overset{\otimes}{,} \overset{2}{B}\}$ for the Poisson case, which is defined as follows:

$$\{\overset{1}{A} \overset{\otimes}{,} \overset{2}{B}\} \overset{\text{def}}{=} \{A_{ij}, B_{kl}\} \otimes E_{ij} \otimes E_{kl}. \tag{8.32}$$

We have proved before (Theorem 1) that the inverse to a Manin matrix is again a Manin matrix, under the condition that M be two sided invertible. Let us give a Poisson version of this theorem Its proof shows, incidentally, the efficiency of matrix notations in calculations.

Theorem 17. *Assume that M is an invertible Poisson–Manin matrix*; *then M^{-1} is again Poisson–Manin matrix.*

Proof. Due to Lemma 28 above it is enough to prove that:

$$\big\{\overset{1}{M}{}^{-1} \overset{\otimes}{,} \overset{2}{M}{}^{-1}\big\} = P\big\{\overset{1}{M}{}^{-1} \overset{\otimes}{,} \overset{2}{M}{}^{-1}\big\}. \tag{8.33}$$

This can be achieved by the following straightforward calculation:

$$\big\{\overset{1}{M}{}^{-1} \overset{\otimes}{,} \overset{2}{M}{}^{-1}\big\} = -\overset{2}{M}{}^{-1}\big\{\overset{1}{M}{}^{-1} \overset{\otimes}{,} \overset{2}{M}{}^{-1}\big\}\overset{2}{M}{}^{-1} = \overset{2}{M}{}^{-1}\overset{1}{M}{}^{-1}\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1} \tag{8.34}$$

$$\text{using “Pyatov's lemma” 28:}\quad \{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\} = P\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\} \text{ we get:} \tag{8.35}$$

$$= \overset{2}{M}{}^{-1}\overset{1}{M}{}^{-1}P\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1} \tag{8.36}$$

$$\text{let us use that } \forall \text{ matrices } A\text{:}\quad \overset{1}{A}P = P\overset{2}{A}, \quad \overset{2}{A}P = P\overset{1}{A} \quad \text{so we get:} \tag{8.37}$$

$$= P\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1}\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1}$$

$$= \text{for matrices with commutating elements } \overset{1}{A}\overset{2}{B} = \overset{2}{B}\overset{1}{A} \tag{8.38}$$

$$= P\overset{2}{M}{}^{-1}\overset{1}{M}{}^{-1}\{\overset{1}{M} \overset{\otimes}{,} \overset{2}{M}\}\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1} = P\big\{\overset{1}{M}{}^{-1} \overset{\otimes}{,} \overset{2}{M}{}^{-1}\big\}. \qquad \square \tag{8.39}$$

Let us finally mention a “curious” corollary, which arises if one uses the same arguments as above in the noncommutative case (that is, not in the Poisson case) and also uses the established fact that M^{-1} is again a Manin matrix.

Lemma 29. *Let M be two-sided invertible Manin matrix. Then:*

$$0 = P\big[\overset{1}{M}{}^{-1}, \overset{2}{M}{}^{-1}\big][\overset{1}{M}, \overset{2}{M}] = P[\overset{1}{M}, \overset{2}{M}]\overset{1}{M}{}^{-1}\overset{2}{M}{}^{-1}[\overset{1}{M}, \overset{2}{M}]. \tag{8.40}$$

9. Conclusion and open questions

Let us make concluding remarks and mention open problems.

In the present paper we considered a class of matrices with noncommutative entries and demonstrated that most of the linear algebra properties can be transferred to this class. We refer to [12,16, 17,19,20,102] for applications and related issues.

The natural question is whether one can extend such theory to the other classes of noncommutative matrices. We already have proved results on q-analogs [20] and more general matrices appearing in Manin's framework, and we hope to publish this in some future. However what seems to be unclear – what can be applications of the "noncommutative endomorphisms" of general algebras, what it can give for studies of Sklyanin algebras or Calabi–Yau algebras [46]?

Although Manin's framework is quite general, it does not seem to cover all the examples of noncommutative matrices which appear in applications and for which some sporadic interesting results has been already obtained.

First class of examples comes from the theory of quantum integrability – almost all integrable systems have Lax matrices, and so their quantum versions provides matrices with noncommutative entries. We expect that linear algebra can be developed in all these cases and it will have important applications [18]; however it is not so clear how to tackle this problem.

Moreover, related questions appear in quantum group and Lie algebra theory: $gl(n)$-related matrices in the non-vector representations [65,101]; quantum groups for non-$gl(n)$-case; twisted current algebras and twisted Yangians (e.g. [82]); reflection equation and more general quadratic algebras L. Freidel and J.M. Maillet [38]. Let us mention quite an interesting matrix related to symmetric group S_n appearing in M. Gould [49, page 1], and P. Biane [7] from completely different points of view, [49] also contains generalizations to the more general finite groups.

The diversity of interesting noncommutative matrices suggest that it might natural not to work case by case, but rather ask a general question: given a matrix with noncommutative entries is it possible to understand whether its proper determinant does exist? If yes, how to develop linear algebra?

So we see that there is a field for the further research which concerns generalization of Manin matrices. Let us finally mention some more concrete questions related to Manin matrices themselves.

9.1. Tridiagonal matrices and duality in Toda system

Let us recall a matrix identity for tridiagonal matrices and rise a question on its extension to Manin matrices. The identity is very well known in integrability theory, and implies that the classical Toda system has two different Lax representations: one by $n \times n$ matrices and another by 2×2 matrices. In the language of integrability theory our question is about quantization of this identity.

Let us consider commutative variables x_i and p_i; consider the following matrices:

$$\mathcal{L}_{n\times n}(v) = \begin{pmatrix} -p_1 & 1 & \dots & 0 & v^{-1}e^{x_{1n}} \\ e^{x_{21}} & -p_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -p_{n-1} & 1 \\ v & 0 & \dots & e^{x_{n,n-1}} & -p_n \end{pmatrix}, \qquad x_{jk} \equiv x_j - x_k, \tag{9.1}$$

$$L_{2\times 2}(u) = \begin{pmatrix} u+p_n & -e^{x_n} \\ e^{-x_n} & 0 \end{pmatrix}\begin{pmatrix} u+p_{n-1} & -e^{x_{n-1}} \\ e^{-x_{n-1}} & 0 \end{pmatrix}\dots\begin{pmatrix} u+p_1 & -e^{x_1} \\ e^{-x_1} & 0 \end{pmatrix}. \tag{9.2}$$

The following identity is well known in integrability theory (e.g. E.K. Sklyanin [105, Section 2.50]), and is also known in the other fields of mathematical physics [84], Theorem 1 (there its generalization for block tridiagonal matrices has been found).

Lemma 30. *The following holds true*:

$$(-1)^{n-1}\det\bigl(u-\mathcal{L}_{n\times n}(v)\bigr)=v^2\,\det^{\mathrm{col}}\bigl(1-v^{-1}L_{2\times 2}(u)\bigr). \tag{9.3}$$

Now consider the quantum case, i.e. consider noncommuting variables $\hat{p}_i$ and $\hat{x}_i$, such that they satisfy the Heisenberg standard commutation relations: $[\hat{p}_i,\hat{x}_j]=\delta_{ij}$, $[\hat{p}_i,\hat{p}_j]=[\hat{x}_i,\hat{x}_j]=0$. Consider the matrices $\widehat{\mathcal{L}}_{n\times n}(u)$, $\widehat{L}_{2\times 2}(u)$ defined by the same formulas as above substituting $\hat{x}_i$ for x_i and respectively $\hat{p}_i$ for p_i.

One can see directly (see also [16], Section 3.2), that

$$e^{-\partial_u}\widehat{L}_{2\times 2}(u)\quad\text{is a Manin matrix.} \tag{9.4}$$

Question. Is it possible to find an appropriate definition for "$\det(u-\widehat{\mathcal{L}}_{n\times n}(e^{\partial_u}))$" such that:

$$(-1)^{n-1}\text{“}\det\bigl(u-\widehat{\mathcal{L}}_{n\times n}\bigl(e^{\partial_u}\bigr)\bigr)\text{”}=e^{2\partial_u}\det^{\mathrm{col}}\bigl(1-e^{-\partial_u}\widehat{L}_{2\times 2}(u)\bigr)? \tag{9.5}$$

The matrix $u-\widehat{\mathcal{L}}_{n\times n}(e^{\partial_u})$ does not seem to be related to Manin matrices at least in any simple way. So solution to this problem may lead to the development of linear algebra for noncommutative matrices beyond Manin's case, which is highly desired.

Remark 41. For readers who are not familiar with Lax matrices let us add the following brief remark (see also [16], Section 3). Lax matrices $L(z)$ should satisfy several properties, perhaps the most basic of which being that their characteristic polynomial $\det^{\mathrm{col}}(\lambda-L(z))=\sum_{ij}H_{ij}z^i\lambda^j$ should produce a complete set of Liouville integrals of motion. This means that $\{H_{ij},H_{kl}\}=0$ and any further integral of motion is a function of H_{ij}. Lax matrices have been found for the majority of integrable systems. One integrable system may have several Lax matrices. Matrices $L_{2\times 2}(u)$, $\mathcal{L}_{n\times n}(u)$ are such two examples of Lax matrices for the same system (the Toda classical integrable system). In the quantum case it is quite natural to look for a kind of determinantal formula: "$\det^{\mathrm{col}}(\hat{\lambda}-L(\hat{z}))$" to produce all quantum integrals of motion: $[\widehat{H}_{ij},\widehat{H}_{kl}]=0$, and, possibly, to satisfy other important properties (see [18]). Formula $\det^{\mathrm{col}}(1-e^{-\partial_u}\widehat{L}_{2\times 2}(u))$ is such a formula for quantum Toda system.

9.2. Fredholm type formulas

In the commutative case the following formulas can be found in [15]. Fredholm's formulas for the solution of an integral equation is a particular case of them [15, Section 4].

Let $\mathbb{C}[x_1,x_2,\ldots,x_n]$ be the algebra of polynomials. Let $\mathbf{C}(x_1,x_2,\ldots)$ be the operator of multiplication by the polynomial $C(x_1,x_2,\ldots)$; Let $\mathbf{A}(\partial_{x_1},\partial_{x_2},\ldots)$ be a polynomial in the operators $\partial_{x_1}=\frac{\partial}{\partial x_1}$, $\partial_{x_2}=\frac{\partial}{\partial x_2},\ldots.$

Conjecture 1. Consider $n\times n$ Manin matrix M, assume that $[M_{ij},x_k]=0$ and define its action on $\mathbb{C}[x_1,x_2,\ldots,x_n]$ in a standard way: $x_i\to\sum_j M_{ij}x_j$ and $M(x_{i_1}\ldots x_{i_k})=M(x_{i_1})\ldots M(x_{i_k})$. Then:

$$\begin{aligned}&\mathop{Tr}_{\mathbb{C}[x_1,x_2,\ldots,x_n]}\bigl(M\mathbf{A}(\partial_{x_1},\partial_{x_2},\ldots.)\mathbf{C}(x_1,x_2,\ldots)\bigr)\\&\quad=(\mathop{Tr}_{\mathbb{C}[x_1,x_2,\ldots,x_n]}M)\left\langle A(\partial_{x_1},\partial_{x_2},\ldots.)\,\middle|\,\frac{1}{1-M}C(x_1,x_2,\ldots)\right\rangle.\end{aligned} \tag{9.6}$$

Here the pairing $\langle \ldots | \ldots \rangle$ is defined by the formula:

$$\left\langle (\partial_{x_1})^{i_1} \ldots (\partial_{x_n})^{i_n} \,\middle|\, (x_1)^{j_1} \ldots (x_n)^{j_n} \right\rangle = \delta_{i_1}^{j_1} \ldots \delta_{i_n}^{j_n} i_1! \ldots i_n!.$$

The trace of the operator M: $\mathbb{C}[x_1, x_2, \ldots] \to \mathbb{C}[x_1, x_2, \ldots]$ is the sum of diagonal elements in the natural basis $(x_1)^{i_1}(x_2)^{i_2} \ldots (x_k)^{i_k}$.

Analog formulas are conjectured to hold for the anticommuting variables $\xi_i \xi_j = -\xi_j \xi_i$:

Conjecture 2.

$$\begin{aligned} &\operatorname*{Tr}_{\Lambda[\xi_1,\xi_2,\ldots]} \big(M\mathbf{A}(\partial_{\xi_1}, \partial_{\xi_2}, \ldots)\mathbf{C}(\xi_1, \xi_2, \ldots)\big) \\ &\quad = (\operatorname*{Tr}_{\Lambda[\xi_1,\xi_2,\ldots]} M)\left\langle A(\partial_{\xi_1}, \partial_{\xi_2}, \ldots) \,\middle|\, \frac{1}{1+M} C(\xi_1, \xi_2, \ldots) \right\rangle. \end{aligned} \tag{9.7}$$

Corollary 9. *Let* $\mathbf{v} = \sum_k v_k x_k$, $\mathbf{w} = \sum_k w_k \partial_{x_k}$, $C(e^{\mathbf{v}}) : \mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[x_1, x_2, \ldots, x_n]$ *operator of multiplication on* $(e^{\mathbf{v}})$ *and* $A(e^{\mathbf{w}}) : \mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[x_1, x_2, \ldots, x_n]$ *exponential of differentiation operator. Then*

$$\frac{Tr_{\mathbb{C}[x_1,x_2,\ldots,x_n]}(MA(e^{\mathbf{w}})C(e^{\mathbf{v}}))}{Tr_{\mathbb{C}[x_1,x_2,\ldots,x_n]}(M)} = e^{\langle \mathbf{w} | \frac{1}{1-M} \mathbf{v} \rangle} \tag{9.8}$$

(*see [15], Formula 3.4*).

9.3. Tensor operations, immanants, Schur functions …

In the commutative case one can consider tensor powers V^{μ} of $V = \mathbb{C}^n$ and corresponding tensor powers M^{μ} of any matrix M, indexed by a Young diagram μ. Can one extend this to the case of Manin matrices?

The obvious problem is the following: symmetric powers of a Manin matrix M can be defined by the right action on x_i, while antisymmetric powers by the left action on ψ_i. So it is not clear whether the natural way to mix left and right actions exists or not.

Schur functions $S(\lambda_1, \ldots, \lambda_n)$ can be considered as traces $Tr(M^{\mu})$, where λ_i are eigenvalues of M. They satisfy plenty of relations. Can one generalize them to the case of Manin matrices? $Tr(M^{\mu})$ are written explicitly in terms of sum of principal μ-immanants of M $(\sum_{i_1,i_2,\ldots} \sum_{\sigma \in S_n} \mu(\sigma) \prod M_{i_p,\sigma(i_p)}$, where $\mu(\sigma)$ is a character of irrep of S_N corresponding to Young diagram μ). The determinant and permanent are particular cases of immanants. The problem above show up itself again: the permanent was defined by the row-expansion, while the determinant via column expansion. One can try to consider symmetrized immanants. (For Manin matrices symmetrized-determinant equals to column-determinant and symmetrized-permanent equals to row-permanent, since determinant (permanent) behaves well under permutations of columns (rows)).

Progress in this question may be applied to quantum immanants theory by A. Okounkov, G. Olshansky [92–94] (see also [81]) and to the related so-called "fusion" procedure in quantum integrable systems theory (e.g. V. Kazakov, A. Sorin, A. Zabrodin [63], D. Gurevich, P. Pyatov, P. Saponov [53]) via applications to examples of Manin matrices considered in [16].

Noncommutative Schur functions were proposed in [44]. Is this recipe related to symmetrized (or whatever) immanants of Manin matrices? Can one relate these Schur functions to quantum immanants from [92] via specialization to a Manin matrix defined by the first formula of (3.41) page 257?

These questions seems to us quite important, but we have not analyzed them yet. Let us finally list some more problems that might be of interest:

- Is there something interesting about the co-product of $\sigma_k, S_k, Tr(M^k)$? (Hopf algebra structures are useful in symmetric function theory ([44] and references therein), but the natural co-product for Manin matrices is different from that used therein.)
- Let us consider the algebra generated by Manin's relations $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$, and consider the moduli space of all its k-dimensional representations (i.e. just the set of $k \times k$ matrices A_{ij} satisfying the relations above, modulo conjugation). It is some manifold (or orbifold)? What can be said about it? That question might be of some interest since in a particular case it includes the "commuting variety": $A, B : [A, B] = 0$, which is a subject of intensive research.
- Concerning the algebra generated by M_{ij} with the only relations $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$, it seems also little to be known. Are their left or right zero divisors? If no - can one embed it into some field of fractions? What is its Poincarè series with respect to the natural grading ?
- Recently some non-linear algebra of multi-index multi-matrices began to emerge [2], where appropriate resultants play role of the determinant. It might be interesting to obtain some non-commutative generalizations of their results in the spirit of the present paper.
- Can one successfully study properties of *random* Manin matrices?

Acknowledgments

G.F. acknowledges support from the ESF programme MISGAM, and the Marie Curie RTN ENIGMA. The work of A.C. has been partially supported by the Russian President Grant MK-5056.2007.1, grant of Support for the Scientific Schools NSh-3035.2008.2, RFBR grant 08-02-00287a, the ANR grant GIMP (Geometry and Integrability in Mathematics and Physics). He acknowledges support and hospitality of Angers and Poitiers Universities. The work of V.R. has been partially supported by the grant of Support for the Scientific Schools NSh-3036.2008.2, RFBR grant 06-02-17382, the ANR grant GIMP (Geometry and Integrability in Mathematics and Physics) and INFN-RFBR project "Einstein". He acknowledges support and hospitality of SISSA (Trieste). The authors are grateful to D. Talalaev, A. Molev, A. Smirnov, A. Silantiev, V. Retakh and D. Gurevich for multiple stimulating discussions, to P. Pyatov for sharing with us his unpublished results, pointing out to the paper [39] and for multiple stimulating discussions. To Yu. Manin for his interest in this work and stimulating discussions.

References

[1] K. Adjamagbo, Panorama de la theorique des determinants sur un anneau non commutatif, Bull. Sci. Math. 117 (1993) 401–420.

[2] E. Akhmedov, V. Dolotin, A. Morozov, Comment on the surface exponential for tensor fields, http://arxiv.org/abs/hep-th/0504160;
V. Dolotin, A. Morozov, Introduction to Non-Linear Algebra, World Sci. Publ., Hackensack, NJ, 2007, 269 pp. http://arxiv.org/abs/hep-th/0609022;
A. Morozov, Sh. Shakirov, Analogue of the identity Log Det = Trace Log for resultants, http://arxiv.org/abs/0804.4632.

[3] O. Babelon, M. Talon, Riemann surfaces, separation of variables and classical and quantum integrability, Phys. Lett. A 312 (2003) 71–77.

[4] T. Banica, J. Bichon, B. Collins, Quantum permutation groups: A survey, in: Noncommutative Harmonic Analysis with Applications to Probability, in: Banach Center Publ., vol. 78, Polish Acad. Sci., Warsaw, 2007, pp. 13–34.

[5] E.H. Bareiss, Sylvester's identity and multistep integer-preserving Gaussian elimination, Math. Comp. 22 (1968) 565–578.

[6] A. Berenstein, A. Zelevinsky, String bases for quantum groups of type A_r, in: I.M. Gelfand Seminar, in: Adv. Sov. Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 51–89.

[7] P. Biane, Approximate factorization and concentration for characters of symmetric groups, Int. Math. Res. Not. 2001 (2001) 179–192.

[8] A. Bracken, H. Green, Vector operators and a polynomial identity for SO(n), J. Math. Phys. 12 (1971) 2099–2106.

[9] D. Bressoud, Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture, Cambridge University Press, 1999.

[10] P. Caldero, R. Marsh, A multiplicative property of quantum flag minors II, J. London Math. Soc. (2) 69 (2004) 608–622.

[11] A. Capelli, Uber die Zuruckfuhrung der Cayley'schen Operation Ω auf gewohnliche Polar-Operationen, Math. Ann. 29 (1887) 331–338.

[12] S. Caracciolo, A. Sportiello, A.D. Sokal, Noncommutative determinants, Cauchy–Binet formulae, and Capelli-type identities. I. Generalizations of the Capelli and Turnbull identities, arXiv:0809.3516.

[13] P. Cartier, D. Foata, Problemes combinatoires de commutation et rearrangements, Lecture Notes in Math., vol. 85, Springer-Verlag, Berlin, Heidelberg, 1969.

[14] V. Chari, A. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.

[15] A. Chervov, Traces of creation-annihilation operators and Fredholm's formulas, Funct. Anal. Appl. 32 (1998) 71–74.
[16] A. Chervov, G. Falqui, Manin matrices and Talalaev's formula, J. Phys. A 41 (2008) 194006.
[17] A. Chervov, A. Molev, On higher order Sugawara operators, arXiv:0808.1947.
[18] A. Chervov, D. Talalaev, Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence, hep-th/0604128.
[19] A. Chervov, G. Falqui, L. Rybnikov, Limits of Gaudin algebras, quantization of bending flows, Jucys–Murphy elements and Gelfand–Tsetlin bases, arXiv:0710.4971;
A. Chervov, G. Falqui, L. Rybnikov, Limits of Gaudin systems: Classical and quantum cases, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009) 029, 17 pp.
[20] A. Chervov, G. Falqui, V. Rubtsov, A. Silantiev, in press.
[21] E.E. Demidov, Yu.I. Manin, E.E. Mukhin, D.V. Zhdanovich, Non-standard quantum deformations of GL(n) and constant solutions of the Yang–Baxter equation, Progr. Theoret. Phys. Suppl. 102 (1990) 203–218.
[22] J. Dieudonné, Les determinantes sur un corps non-commutatif, Bull. Soc. Math. France 71 (1943) 27–45.
[23] C.L. Dodgson, Condensation of determinants, Proc. R. Soc. London 15 (1866) 150–155.
[24] M. Domokos, Cayley–Hamilton theorem for 2×2 matrices over the Grassmann algebra, in: Ring Theory, Miskolc, 1996, J. Pure Appl. Algebra 133 (1998) 69–81.
[25] M. Domokos, T.H. Lenagan, The traces of quantum powers commute, arXiv:math.QA/0302063.
[26] B. Enriquez, V. Rubtsov, Hitchin systems, higher Gaudin operators and r-matrices, Math. Res. Lett. 3 (1996) 343–357.
[27] B. Enriquez, V. Rubtsov, Commuting families in skew fields and quantization of Beauville's fibration, Duke Math. J. 119 (2003) 197–219.
[28] P. Etingof, I. Pak, An algebraic extension of the MacMahon Master theorem, Proc. Amer. Math. Soc. 136 (2008) 2279–2288.
[29] H. Ewen, O. Ogievetsky, J. Wess, Quantum matrices in two dimensions, Lett. Math. Phys. 22 (1991) 297–305.
[30] L.D. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin, 1987, x+592 pp.
[31] L.D. Faddeev, L.A. Takhdadjan, E.K. Sklyanin, Quantum inverse problem method I, Theoret. Math. Phys. 40 (1979) 194–220.
[32] B. Feigin, E. Frenkel, N. Reshetikhin, Gaudin model, Bethe Ansatz and critical level, Comm. Math. Phys. 166 (1994) 27–62.
[33] R. Fioresi, Commutation relations among generic quantum minors in $\mathcal{O}_q(M_n(k))$, J. Algebra 280 (2004) 655–682.
[34] H. Flaschka, J. Millson, Bending flows for sums of rank one matrices, Canad. J. Math. 57 (2005) 114–158.
[35] D. Foata, A noncommutative version of the matrix inversion formula, Adv. Math. 31 (1979) 330–349.
[36] D. Foata, G.-H. Han, A basis for the right quantum algebra and the "1 = q" principle, J. Algebraic Combin. 27 (2008) 163–172.
[37] D. Foata, D. Zeilberger, Combinatorial proofs of Capelli's and Turnbull's identities from classical invariant theory, Electron. J. Combin. 1 (1994) R1.
[38] L. Freidel, J.M. Maillet, Quadratic algebras and integrable systems, Phys. Lett. B 262 (1991) 278–284.
[39] S. Garoufalidis, T.T.Q. Le, D. Zeilberger, The quantum MacMahon Master theorem, Proc. Natl. Acad. Sci. USA 103 (2006) 13928–13931 (electronic).
[40] M. Gaudin, Diagonalisation d'une classe d'hamiltoniens de spin, J. Phys. 37 (1976) 1087–1098.
[41] I.M. Gelfand, V.S. Retakh, Determinants of matrices over non commutative rings, Funct. Anal. Appl. 25 (1991) 91–102.
[42] I.M. Gelfand, V.S. Retakh, Theory of noncommutative determinants, and characteristic functions of graphs, Funct. Anal. Appl. 26 (1992) 231–246.
[43] I.M. Gelfand, V.S. Retakh, Quasideterminants, I, Selecta Math. 3 (1997) 517–546.
[44] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.
[45] I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, Quasideterminants, Adv. Math. 193 (2005) 56–141.
[46] V. Ginzburg, Calabi–Yau algebras, arXiv:math/0612139.
[47] K.R. Goodearl, Commutation relations for arbitrary quantum minors, Pacific J. Math. 228 (2006) 63–102.
[48] M.D. Gould, Characteristic identities for semi-simple Lie algebras, J. Aust. Math. Soc. B 26 (1985) 257–283.
[49] M. Gould, Identities and characters for finite groups, J. Phys. A 20 (1987) 2657–2665.
[50] M.D. Gould, R.B. Zhang, A.J. Bracken, Generalized Gelfand invariants of quantum groups, J. Phys. A 24 (1991) 937–943.
[51] M.D. Gould, R.B. Zhang, A.J. Bracken, Generalized Gelfand invariants and characteristic identities for quantum groups, J. Math. Phys. 32 (1991) 2298–2303.
[52] H.S. Green, Characteristic identities for generators of $GL(n)$; $O(n)$ and $SP(n)$, J. Math. Phys. 12 (1971) 2106–2113.
[53] D. Gurevich, P. Pyatov, P. Saponov, The Cayley–Hamilton theorem for quantum matrix algebras of $GL(m|n)$ type, St. Petersburg Math. J. 17 (2006) 119–135.
[54] P.H. Hai, M. Lorenz, Koszul algebras and the quantum MacMahon Master theorem, Bull. Lond. Math. Soc. 39 (2007) 667–676.
[55] M.J. Hopkins, A.I. Molev, A q-analogue of the centralizer construction and skew representations of the quantum affine algebra, SIGMA Symmetry Integrability Geom. Methods Appl. 2 (2006), Paper 092, 29 pp. (electronic).
[56] R. Howe, Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989) 539–570; Erratum: Trans. Amer. Math. Soc. 318 (1990) 823;
R. Howe, T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann. 290 (1991) 565–619.
[57] A. Isaev, O. Ogievetsky, P. Pyatov, On quantum matrix algebras satisfying the Cayley–Hamilton–Newton identities, J. Phys. A 32 (1999) L115–L121.
[58] M. Itoh, Explicit Newton's formulas for gln, J. Algebra 208 (1998) 687–697.
[59] M. Itoh, A Cayley–Hamilton theorem for the skew Capelli elements, J. Algebra 242 (2001) 740–761;
M. Itoh, Capelli elements for the orthogonal Lie algebras, J. Lie Theory 10 (2000) 463–489.

[60] P.D. Jarvis, D.S. McAnally, Generalized characteristic identities and applications, in: Confronting the Infinite, Adelaide, 1994, World Sci. Publ., River Edge, NJ, 1995, pp. 215–224.
[61] I. Kantor, I. Trishin, On the Cayley–Hamilton equation in the supercase, Comm. Algebra 27 (1999) 233–259;
I. Trishin, On representations of the Cayley–Hamilton equation in the supercase, Comm. Algebra 27 (1999) 261–287.
[62] T. Kato, Perturbation Theory for Linear Operators, Grundlehren Math. Wiss., vol. 132, Springer-Verlag, 1966.
[63] V. Kazakov, A. Sorin, A. Zabrodin, Supersymmetric Bethe Ansatz and Baxter equations from discrete Hirota dynamics, Nuclear Phys. B 790 (2008) 345–413.
[64] A.C. Kelly, T.H. Lenagan, L. Rigal, Ring theoretic properties of quantum grassmannians, J. Algebra Appl. 3 (2004) 9–30.
[65] A.A. Kirillov, Introduction to family algebras, Mosc. Math. J. 1 (2001) 49–63.
[66] M. Konvalinka, A generalization of Foata's fundamental transformation and its applications to the right-quantum algebra, math.CO/0703203.
[67] M. Konvalinka, Non-commutative Sylvester's determinantal identity, Electron. J. Combin. 14 (2007), Research Paper 42, 29 pp. (electronic).
[68] M. Konvalinka, I. Pak, Non-commutative extensions of the MacMahon Master Theorem, Adv. Math. 193 (2005) 56–141.
[69] D. Krob, B. Leclerc, Minor identities for quasi-determinants and quantum determinants, Comm. Math. Phys. 169 (1995) 1–23.
[70] A. Lauve, Quantum- and quasi-Plücker coordinates, J. Algebra 296 (2006) 440–461.
[71] A. Lauve, E.J. Taft, A class of left quantum groups modeled after SLq(r), J. Pure Appl. Algebra 208 (2007) 797–803.
[72] B. Leclerc, A. Zelevinsky, Quasicommuting families of quantum Plücker coordinates, in: Kirillov's Seminar on Representation Theory, in: Amer. Math. Soc. Transl. Ser. 2, vol. 181, Amer. Math. Soc., Providence, RI, 1998, pp. 85–108.
[73] A.J. Macfarlane, H. Pfeiffer (Cambridge), On characteristic equations, trace identities and Casimir operators of simple Lie algebras, J. Math. Phys. 41 (2000) 3192–3225; Erratum: J. Math. Phys. 42 (2001) 977.
[74] J.-M. Maillet, Lax equations and quantum groups, Phys. Lett. B 245 (1990) 480–486.
[75] Yu. Manin, Some remarks on Koszul algebras and quantum groups, Ann. Inst. Fourier 37 (1987) 191–205.
[76] Yu. Manin, Quantum Groups and Non Commutative Geometry, Universite de Montreal, Centre de Recherches Mathematiques, Montreal, QC, 1988, 91 pp.
[77] Yu. Manin, Topics in Non Commutative Geometry, M.B. Porter Lectures, Princeton University Press, 1991, 164 pp.
[78] Yu. Manin, Notes on quantum groups and quantum de Rham complexes, Theoret. and Math. Phys. 92 (1992) 997–1023.
[79] A.I. Molev, Yangians and their applications, in: Handbook of Algebra, vol. 3, North-Holland, Amsterdam, 2003, pp. 907–959.
[80] A.I. Molev, Skew representations of twisted Yangians, Selecta Math. (N.S.) 12 (2006) 1–38.
[81] A. Molev, Yangians and Classical Lie Algebras, Math. Surveys Monogr., vol. 143, Amer. Math. Soc., Providence, RI, 2007.
[82] A.I. Molev, E. Ragoucy, Symmetries and invariants of twisted quantum algebras and associated Poisson algebras, Rev. Math. Phys. 20 (2008) 173–198.
[83] A. Molev, M. Nazarov, G. Olshanski, Yangians and classical Lie algebras, Russian Math. Surveys 51 (1996) 205–282.
[84] L.G. Molinari, Determinants of block tridiagonal matrices, Linear Algebra Appl. 429 (2008) 2221–2226.
[85] G. Muhlbach, M. Gasca, A Generalization of Sylvester's identity on determinants and some applications, Linear Algebra Appl. 66 (1985) 221–234.
[86] T. Muir, The Theory of Determinants in the Historical Order of Development.
[87] E. Mukhin, V. Tarasov, A. Varchenko, A generalization of the Capelli identity, math.QA/0610799.
[88] M. Nazarov, Quantum Berezinian and the classical capelli identity, Lett. Math. Phys. 21 (1991) 123–131.
[89] M. Nazarov, V. Tarasov, Yangians and Gelfand–Zetlin bases, Publ. Res. Inst. Math. Sci. 30 (1994) 459–478.
[90] D. O'Brien, A. Cant, A. Carey, On characteristic identities for Lie algebras, Ann. Inst. H. Poincare (A) Phys. Theor. 26 (1977) 405–429.
[91] O. Ogievetsky, A. Vahlas, The relation between two types of characteristic equations for quantum matrices, Lett. Math. Phys. 65 (2003) 49–57.
[92] A. Okounkov, Quantum immanants and higher Capelli identities, Transform. Groups 1 (1996) 99–126.
[93] A. Okounkov, Young basis, Wick formula and higher Capelli identities, Int. Math. Res. Not. 17 (1996) 817–839.
[94] A. Okounkov, G. Olshansky, Shifted Schur functions, St. Petersburg Math. J. 9 (1998) 239–300.
[95] A.M. Perelomov, V.S. Popov, Casimir operators for $U(n)$ and $SU(n)$, Sov. J. Nucl. Phys. 3 (1966) 676–680.
[96] C. Procesi, A formal inverse to the Cayley–Hamilton theorem, J. Algebra 107 (1987) 63–74.
[97] P. Pyatov, P. Saponov, Characteristic relations for quantum matrices, J. Phys. A 28 (1995) 4415–4421.
[98] N.Yu. Reshetikhin, L.A. Takhtadzhyan, L.D. Faddeev, Quantization of Lie groups and Lie algebras (in English), Russian original: Leningr. Math. J. 1 (1990) 193–225; translation from: Algebra Anal. 1 (1989) 178–206.
[99] S. Rodriguez-Romo, E. Taft, Some quantum-like Hopf algebras which remain noncommutative when $q = 1$, Lett. Math. Phys. 61 (2002) 41–50.
[100] S. Rodriguez-Romo, E. Taft, A left quantum group, J. Algebra 286 (2005) 154–160.
[101] N. Rozhkovskaya, Braided central elements, arXiv:math/0510226.
[102] V. Rubtsov, A. Silantiev, D. Talalaev, Manin matrices, elliptic commuting families and characteristic polynomial of quantum gl_n elliptic Gaudin model, in press.
[103] J.S. Scott, Quasi-commuting families of quantum minors, J. Algebra 290 (2005) 204–220.
[104] S. Sergeev, Quantum matrices of the coefficients of a discrete linear problem (in Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 269 (2000) 292–307, 370–371; translation in: J. Math. Sci. (N. Y.) 115 (2003) 2049–2057.
[105] E.K. Sklyanin, Bäcklund transformations and Baxter's Q-operator, in: Integrable Systems: From Classical to Quantum, Montréal, QC, 1999, in: CRM Proc. Lecture Notes, vol. 26, Amer. Math. Soc., Providence, RI, 2000, pp. 227–250.
[106] Z. Skoda, Included-row exchange principle for quantum minors, math.QA/0510512.

[107] J. Szigeti, New determinants and the Cayley–Hamilton theorem for matrices over Lie nilpotent rings, Proc. Amer. Math. Soc. 125 (1997) 2245–2254.

[108] J. Szigeti, Cayley–Hamilton theorem for matrices over an arbitrary ring, Serdica Math. J. 32 (2006) 269–276.

[109] K. Takasaki, Integrable systems whose spectral curve is the graph of a function, in: Superintegrability in Classical and Quantum Systems, in: CRM Proc. Lecture Notes, vol. 37, Amer. Math. Soc., Providence, RI, 2004, pp. 211–222.

[110] D. Talalaev, Quantization of the Gaudin system, Funct. Anal. Appl. 40 (2006) 86–91.

[111] H.W. Turnbull, Symmetric determinants and the Cayley and Capelli operators, Proc. Edinb. Math. Soc. (2) 8 (1948) 76–86.

[112] T. Umeda, Newton's formula for gl(n), Proc. Amer. Math. Soc. 126 (1998) 3169–3176.

[113] T. Umeda, Application of Koszul complex to Wronski relations for $U(gl_n)$, Comment. Math. Helv. 78 (4) (2003) 663–680.

[114] L. Urrutia, N. Morales, The Cayley–Hamilton theorem for supermatrices, J. Phys. A 27 (1994) 1981–1997;
D.E. Berenstein, L.F. Urrutia, The equivalence between the Mandelstam and the Cayley–Hamilton identities, J. Math. Phys. 35 (1994) 1922–1930.

[115] S. Wang, Quantum symmetry groups of finite spaces, Comm. Math. Phys. 195 (1998) 195–211.

[116] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press Inc., New York, 1988.

[117] D. Zeilberger, Reverend Charles to the aid of Major Percy and Fields-Medalist Enrico, Amer. Math. Monthly 103 (1996) 501–502;
D. Zeilberger, Dodgson's determinant-evaluation rule proved by two-timing men and women, arXiv:math/9808079;
A.N.W. Hone, Dodgson condensation, alternating signs and square ice, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 364 (2006) 3183–3198;
R.E. Schwartz, Discrete monodromy, pentagrams, and the method of condensation, arXiv:0709.1264;
K. Said, A. Salem, R. Belgacem, A mathematical proof of Dodgson's algorithm, arXiv:0712.0362.

[118] J.J. Zhang, The quantum Cayley–Hamilton theorem, J. Pure Appl. Algebra 129 (1998) 101–109.