Hamiltonian systems on the ”coupled” curves, Nambu-Poisson mechanics and Fairlie-type integrable systems

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Abstract

We consider some polynomial Poisson structures induced by Nambu brackets and Hamiltonian systems associated with them. The case of Poisson brackets induced by the canonical Nambu structure is discussed in details. We show that it provides in a natural way a Poisson structure on the product of the two algebraic curves. We consider explicitly several types of the integrable systems associated with such a construction. In particular we obtain the "elegant" integrable system of Fairlie and its various generalizations, two particle elliptic and "double elliptic" Calogero system and their 'higher genus' analogs.

1 Introduction

Exact solvability of non-linear systems of ordinary differential equations is connected with a possibility to represent them in Hamiltonian form and find a set of mutually commuting (with respect to the corresponding Poisson structure) integrals of motion.

One of the possible generalizations of the Hamiltonian formalism and Poisson brackets is so-called multi-Hamiltonian (n-Hamiltonian) formalism and Nambu brackets [1], [2] which is an n-ary generalization of Poisson brackets. It permits to write down dynamical equations possessing at once $n-1$ evident first integrals — "Hamiltonians" of the Nambu system under consideration. On the other hand using a Nambu-Poisson structure it is possible to express Poisson brackets and Hamiltonian equations using $n-2$ Hamiltonians involving in the definition of the Nambu dynamics. The resulting Poisson bracket will be degenerated and the chosen $n-2$ Hamiltonians will coincide with the Casimir functions of this bracket. The level surface of these functions are symplectic leaves of the corresponding Poisson structure and one can always restrict the dynamics of all the considered Hamiltonian systems onto them. Hence, this procedure permits one to define a Poisson structure on a wide class of manifolds — on any submanifold of codimension $(n-2)$ in the initial Nambu manifold.

In the present paper we consider a situation with so-called "canonical" Nambu brackets when the initial phase space is $\mathbb{C}^{n+k}$ ($\mathbb{R}^{n+k}$). The corresponding symplectic leaves of the Nambu-Poisson structure have a dimension 2 and, hence, all initial complicated nonlinear dynamical systems in $n+k$ dynamical variables turned out to be simply integrable in the sense of Liouville (after restriction onto all symplectic leaves whose geometry is completely determined by the $n+k-2$ functions taken to be Casimirs). We restrict ourselves to a consideration of the most simple "physical" situation when there are two groups of variables on the initial phase space - n "coordinates" and k "momenta" such that the resulting two-dimensional symplectic leaf coincides with a product of two (complex) one-dimensional manifolds – algebraic curves embedded into the spaces of "coordinates" and "momenta" correspondingly. We call such systems "Hamiltonian systems on the coupled curves".

It turned out that there are a lot of remarkable examples among this class of integrable systems and that they generalize some well-known such as the Euler-Nahm top, the "elegant" Fairlie integrable system [3] as well as their closed "cousins" [**?**]. The Fairlie "elegant" system is associated with an algebraic curve embedded into a linear space by a system of special quadrics. The curve is a covering of a hyperelliptic curve. It is necessary to note that the hyperelliptic curves appear also as auxiliary spectral curves in the other class of the integrable models (see [5], [6],[7] for the general case and [8] for the elliptic case). On the contrary, in the present paper, the corresponding parameters on the curve are not auxiliary but play a role of the dynamical variables. It should be noted also, that a "Nambu" interpretation to the Fairlie systems was known previously (see f.e. [10],[**?**],[4]). In our paper we give a Hamiltonian interpretation to the Fairlie systems. We also obtain a class of integrable generalizations ("doubling") of Fairlie systems which we call "coupled" or "doubled" Fairlie systems[1]. At last, one more integrable generalization of Fairlie systems and "coupled" Fairlie systems is connected with special higher order curves. This generalization seems to be also new and they contain in particular a seven-dimensional generalization of Euler-Nambu top of Fairlie and Ueno ([**?**],[**?**])

Another interesting integrable Hamiltonian system that arising in the framework of our approach is a "non-harmonic oscillator on algebraic curves". It occurs that in the case when the corresponding curve is elliptic it coincides with the two-particle Calogero system, and in other cases could be considered as its higher genus analogs.

The paper organizes in the following way: we start with a general remind of Nambu brackets and the Nambu-Poisson structures. Then we define in obvious manner a notion of the Nambu-Poisson structure associated with a pair of "coupled" curves. We specify some easy but important partial cases of this construction and the appearing integrable Hamiltonian systems...

We shall work basically with smooth (complex or real) (C^{∞}) manifolds. Sometimes we will use also algebraic curves and varieties with underlying holomorphic Poisson or symplectic

[1]Another type of the "doubling" to the Fairlie system — namely its complexification was recently considered in the framework of the Hamiltonian formalism in [9]

brackets.

2 Nambu bracket and Poisson bracket

2.1 General Construction

In this section we will remind the basic facts about Nambu and Nambu-Poisson structures [1], [2]. These structures had appeared in the framework of the Nambu mechanics as a generalization of "classical" Hamiltonian mechanics and the notion of the Nambu bracket which was introduced by Nambu in 1973 [1] as a natural generalization of the Poisson bracket. A Nambu bracket on a manifold M is an antisymmetric n-ary operation:

$$\{., ..., .\} : C^\infty(M)^{\otimes n} \to C^\infty(M)$$

such that the following three properties are valid:

1. The antisymmetry under a transposition $\sigma \in \Sigma_n$:

$$\{f_1, ..., f_n\} = (-1)^{|\sigma|}\{f_{\sigma(1)}, ..., f_{\sigma(n)}, \};$$

2. The coordinate-wise " Leibnitz rule" for any $h \in C^\infty(M)$:

$$\{hf_1, ..., f_n\} = h\{f_1, ..., f_n\} + f_1\{h, ..., f_n\};$$

3. The "fundamental identity" (which replaces the usual Jacobi identity):

$$\{\{f_1, ..., f_n\}, f_{n+1}, ..., f_{2n-1}\} + \{f_n, \{f_1, ..., (\check{f_n}), f_{n+1}\}, f_{n+2}, ..., f_{2n-1}\} + ...$$
$$+\{f_n, ..., f_{2n-2}, \{f_1, ..., f_{n-1}, f_{2n-1}\}\} = \{f_1, ..., f_{n-1}, \{f_n, f_{n+1}, ..., f_{2n-1}\}\}$$

for any $f_1, ., f_{2n-1} \in C^\infty(M)$.

A dynamics on a Nambu manifold is governed by $n-1$ Hamiltonians H_1,..., H_{n-1}:

$$\frac{dx_i}{dt} = \{x_i, H_1, ..., H_{n-1}\},$$

where x_i are local coordinates on M.

Example 1. The most common example of the Nambu structure is the so-called "canonical" Nambu structure on the linear space $\mathbb{C}^n$ ($\mathbb{R}^n$) with the coordinates $x_1, ...x_n$:

$$\{f_1, ..., f_n\} = \mathrm{Jac}(f_1, ..., f_n) = \frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)}. \quad (1)$$

2.2 Nambu-Poisson bracket on the "coupled" curves

As we have mentioned above a Nambu bracket is an n-ary generalization of the Poisson bracket by the very construction. On the other hand it is easy to construct an ordinary Poisson bracket starting with a Nambu structure. Let us consider the simplest example of this construction. Let M be a Nambu manifold with an n-ary bracket $\{., ..., .\}$. Let us fix $n-2$ arbitrary functions $F_1, ...F_{n-2}$ on M and define with their help the following bracket:

$$\{f, g\} \equiv \{f, g, F_1, ..., F_{n-2}\}. \tag{2}$$

It follows easily from the above properties of Nambu structure that the bilinear differential operation (2) is a Poisson bracket. It follows immediately that functions $F_1, ..., F_k$ are Casimir functions of this Poisson structure, i.e. $\{f, F_k\} = 0$ for any function f on M.

Let us now consider the canonical Nambu bracket (1) on the space $\mathbb{C}^n$. Let us fix $n-2$ functions F_k and construct using the formula (2) the corresponding Poisson brackets. It is evident that it provides a symplectic structure on the two-dimensional manifolds $\mathfrak{M}$ -the common level set of the Casimir Functions F_k (symplectic leaf of the Poisson structure (2)).

Now we will pass to the description of the corresponding structure in the most interesting case which will be the basic throughout this note.

Let us now consider the space $\mathbb{C}^{n+k}$ with the coordinates $x_1 = q_1, \ldots, x_n = q_n, x_{n+1} = p_1, \ldots, x_k = p_k$ and the following induced "canonical" Nambu-Poisson bracket (2) defined with the help of $n+k-2$ polynomial functions $F_1(q_1, .., q_n), ..., F_{n-1}(q_1, ..., q_n)$ and $\widetilde{F}_1(p_1, .., p_k), ..., \widetilde{F}_{k-1}(p_1, ..., p_k)$. The common level to each of this two sets of functions defines an algebraic curve embedded in the affine space of the dimension n and k correspondingly. Hence the formula (2) gives us a symplectic structure on the direct product of two algebraic curves.

Let us consider the explicit form of these brackets. Direct calculation gives:

$$\{q_i, p_j\} = \frac{\partial(q_i, F_1...., F_{n-1})}{\partial(q_1, ..., q_n)} \frac{\partial(p_j, \widetilde{F}_1, ..., \widetilde{F}_{k-1})}{\partial(p_1, ..., p_k)} \tag{3}$$

$$\{q_i, q_j\} = \{p_i, p_j\} = 0. \tag{4}$$

Denote by $q = (q_1, \ldots, q_n)$ and by $p = (p_1, \ldots, p_k)$ and by $H = H(q, p)$ a Hamiltonian function on $\mathbb{C}^{n+k}$. Then the corresponding Hamiltonian equations of motion are written as:

$$\frac{dq_i}{dt} = \{q_i, H(q, p)\} = \frac{\partial(q_i, F_1...., F_{n-1})}{\partial(q_1, ..., q_n)} \frac{\partial(H, \widetilde{F}_1, ..., \widetilde{F}_{k-1})}{\partial(p_1, ..., p_k)} \tag{5}$$

$$\frac{dp_i}{dt} = \{p_i, H(q, p)\} = -\frac{\partial(p_i, \widetilde{F}_1, ..., \widetilde{F}_{k-1})}{\partial(p_1, ..., p_k)} \frac{\partial(H, F_1...., F_{n-1})}{\partial(q_1, ..., q_n)} \tag{6}$$

The coordinates q_i and p_i may be viewed as the generalized "coordinates and momentum". There are several interesting examples of such the systems that correspond to the different choices of n and k and different choices of the functions F_i and $\widetilde{F}_j$. We will consider two most interesting cases.

2.3 Case $k = 1$: "Particle" on the algebraic curve

Let us consider the following partial case of this situation when all components of the momentum $p = (p_1, \ldots, p_k)$ are linear functions of a fixed one. In other words, let $k = 1$ and we take $\tilde{F}_i \equiv p_i = p_1 = p$.

In this case we obtain the following bracket on the space $\mathbb{C}^{n+1}$:

$$\{q_i, p\} = \frac{\partial(q_i, F_1 ...., F_{n-1})}{\partial(q_1, ..., q_n)}, \quad \{q_i, q_j\} = 0, \tag{7}$$

where F_i are again Casimir functions.

The equations of motion of a "particle" with the folowing natural Hamiltonian function

$$H = \frac{1}{2}p^2 + V(q_1, ..., q_n)$$

on the corresponding curve has the form:

$$\frac{dq_i}{dt} = p\frac{\partial(q_i, F_1 ...., F_{n-1})}{\partial(q_1, ..., q_n)}, \quad \frac{dp}{dt} = -\frac{\partial(V, F_1 ...., F_{n-1})}{\partial(q_1, ..., q_n)}. \tag{8}$$

These equations generalize canonical Hamiltonian equations onto the case of the algebraic curves. Let us now consider several concrete examples of the algebraic curves and associated integrable Hamiltonian systems.

2.4 Case of the special higher order curves

Let us consider algebraic curve embedded into $\mathbb{C}^n$ by the following equations of the order m:

$$F_1 = 1/m(q_1^m - q_2^m) = c_1, \; F_2 = 1/m(q_2^m - q_3^m) = c_2, \cdots, F_{n-1} = 1/m(q_{n-1}^m - q_n^m) = c_{n-1}. \tag{9}$$

This curve is a covering of a curve defined by the equation

$$y^m(u) = \prod_{i=1}^{n}(u - e_i), \text{ where } y(u) \equiv \prod_{i=1}^{n} q_i, \; q_i^m \equiv (u - e_i).$$

Example 1. The most interesting case of the curves (9) is the case of the second order ($m = 2$) curves:

$$F_1 = 1/2(q_1^2 - q_2^2) = c_1, \; F_2 = 1/2(q_2^2 - q_3^2) = c_2, \cdots, F_{n-1} = 1/2(q_{n-1}^2 - q_n^2) = c_{n-1}. \tag{10}$$

This curve is a covering of a hyperelliptic curve defined by the equation

$$y^2 = \prod_{i=1}^{n}(u - e_i), \text{ where } q_i^2 \equiv (u - e_i).$$

The curve (10) for $n > 3$ is not hyperelliptic. Its genus is equal to $g = (n-3)2^{n-2} + 1$ [7].

Example 2. Let us consider the case $n = 3$. In this case genus of the curve is $g = 1$ and it is elliptic. Indeed (see [8]) equations (10) for $n = 3$ define the embedding of the elliptic curve into $\mathbb{C}^3$. Its uniformization is made by the Weierstrass $\wp$ -function: $u = \wp(x),\ y = 1/2\wp'(x)$. Functions $q_i, i = 1, 2, 3$ are expressed via Jacobi elliptic functions:

$$q_1(x) = \frac{1}{sn(x)},\ q_2(x) = \frac{dn(x)}{sn(x)},\ q_3(x) = \frac{cn(x)}{sn(x)}. \tag{11}$$

Bracket (7) corresponding to the general polynomial curves (9) acquires the form:

$$\{q_i, p\} = \frac{y^{m-1}(u)}{q_i^{m-1}} = \prod_{i \neq k} q_k^{m-1} \quad \{q_i, q_j\} = 0, i, j = 1, \dots, n, \tag{12}$$

or in the coordinates u, p:

$$\{u, p\} = m y^{m-1}(u). \tag{13}$$

Remark 1. In the case $n = 3$, $m = 2$ we obtain that $u(x) = \wp(x)$ is globally defined univalued meromorphic function on torus. In the case $n > 3$ it is not true and all the corresponding expressions have only a local character.

Let us consider two examples of the Hamiltonian systems that are associated with these brackets and correspond to the cases of free-particle and "generalized oscillator" Hamiltonians.

2.4.1 "Free motion on the curve"

Let us consider Hamiltonian system (8) with the Poisson bracket (12) and free Hamiltonian:

$$H = \frac{1}{2}p^2. \tag{14}$$

The Hamiltonian equations of motion that correspond to the evident integral of motion $P = \frac{1}{m}p = \frac{1}{m}\sqrt{2H}$ in the coordinates q_i have the following form:

$$\frac{dq_i}{d\tau} = \prod_{i \neq k} q_k^{m-1}, i = 1, \dots, n. \tag{15}$$

We call this system of the ordinary differential equations "generalized Fairlie system".

In the case $m = 2$ the corresponding system of differential equations acquires the form:

$$\frac{dq_i}{d\tau} = \prod_{i \neq k} q_k, \tag{16}$$

i.e. coincides with the so called "elegant integrable equations" of Fairlie ([3]). In other words we get that the equations of Fairlie coincide with the equations of the free-particle motion on the covering of the hyperelliptic curve.

The solution of the generalized Fairlie system written in the terms of the coordinate u

$$\frac{du}{d\tau} = y^{m-1}(u) \Rightarrow \ \tau(u) = \int \frac{du}{y^{m-1}(u)} \tag{17}$$

provides at the same time canonical conjugated coordinate to the momenta p: $\{p, \tau(u)\} = m$.

Example 3. In the case $n = 3$ of equations (15) we obtain the generalized Nahm top:

$$\frac{dq_1}{d\tau} = q_2^{m-1}q_3^{m-1}, \ \frac{dq_2}{d\tau} = q_1^{m-1}q_3^{m-1}, \ \frac{dq_3}{d\tau} = q_1^{m-1}q_2^{m-1}. \tag{18}$$

In the $m = 2$ case equations (18) coincide with the equations of the Euler-Nahm top:

$$\frac{dq_1}{d\tau} = q_2q_3, \ \frac{dq_2}{d\tau} = q_1q_3, \ \frac{dq_3}{d\tau} = q_1q_2. \tag{19}$$

2.4.2 Unharmonic oscillator on the higher order curves

Let us consider Hamiltonian system (8) with the Poisson bracket (12) and the special polynomial Hamiltonian of the generalized unharmonic oscillator:

$$H = \frac{1}{2}p^2 + \frac{1}{n}\sum_{i=1}^{n} q_i^m. \tag{20}$$

The Hamiltonian equations in the local coordinates u,p have the following form:

$$\frac{du}{dt} = 2mpy^{m-1}(u), \quad \frac{dp}{dt} = -my^{m-1}(u) \tag{21}$$

One can easily integrate these equations:

$$\frac{du}{dt} = 2m\sqrt{(2E-u)}y^{m-1}(u) \Rightarrow \ t = \int \frac{du}{2my^{m-1}(u)\sqrt{(2E-u)}}, \tag{22}$$

where we took into account on the trajectories $H = 1/2p^2 + u = E$ is a constant of motion and put for convenience that $\sum\limits_{i=1}^{n} e_i = 0$.

Example 4. In the case of the Hamiltonian systems on the second order curves ($m = 2$) we obtain the Hamiltonian of n-component harmonic oscillator living on the curve (10):

$$H = \frac{1}{2}p^2 + \frac{1}{n}\sum_{i=1}^{n} q_i^2 = \frac{1}{2}p^2 + u. \tag{23}$$

Remark 2. In the case of the rational degeneration $e_i \to 0$ and $n > 1$ this system does not go to the usual oscillator because in this case also bracket (12) is not canonical. In the case $n = 1$ bracket (12) is canonical and our system coincide with the usual oscillator.

In the elliptic $n = 3$ case corresponding Hamiltonian (23) acquires the form:

$$H = 1/2(p^2 + \wp(x))$$

and coincide with two particle Calogero system. That is why systems with the Hamiltonian (23) may be viewed as a direct "hyperelliptic" generalization of two particle Calogero system.

2.5 Case $k = n$: Hamiltonian systems on the "doubled" curves

Let us again consider the "canonical" Nambu-Poisson brackets (2) defined with the help of $n-1$ polynomial functions $F_1, \ldots, F_{n-1}$ taking them in "two-fold": once as the polynomials in the coordinates $F_1(q_1,..,q_n),..., F_{n-1}(q_1,...,q_n)$ and in other hand, as the functions on momenta: $F_1(p_1,..,p_n),..., F_{n-1}(p_1,...,p_n)$ i.e. the case when the corresponding functions are equal $F_i = \widetilde{F}_i, i = 1, \ldots, n-1$.

The explicit form of these Nambu-Poisson "double" brackets are given by the direct computation:

$$\{q_i, p_j\} = \frac{\partial(q_i, F_1...., F_{n-1})}{\partial(q_1, ..., q_n)} \frac{\partial(p_j, F_1, ..., F_{n-1})}{\partial(p_1, ..., p_n)} \tag{24}$$

$$\{p_i, p_j\} = \{q_i, q_j\} = 0. \tag{25}$$

2.5.1 Special "doubled" higher order curves.

Let us consider the case of the two identical algebraic curves embedded into $\mathbb{C}^{2n}$ by the following equations of the order m:

$$F_1 = 1/m(q_1^m - q_2^m) = c_1,\ F_2 = 1/m(q_2^m - q_3^m) = c_2, \cdots, F_{n-1} = 1/m(q_{n-1}^m - q_n^m) = c_{n-1}, \tag{26}$$

$$\widetilde{F}_1 = 1/m(p_1^m - p_2^m) = c_1',\ \widetilde{F}_2 = 1/m(p_2^m - p_3^m) = c_2', \cdots, \widetilde{F}_{n-1} = 1/m(p_{n-1}^m - p_n^m) = c_{n-1}'. \tag{27}$$

This curve is a covering of a curve defined by the equation

$$y^m(u) = \prod_{i=1}^n (u - e_i), \widetilde{y}^m(v) = \prod_{i=1}^n (v - e_i') \text{ where } q_i^m \equiv (u - e_i), p_i^m \equiv (v - e_i').$$

In this case the Poisson-Nambu bracket has the form:

$$\{q_i, p_j\} = \frac{y^{m-1}}{q_i^{m-1}} \frac{\widetilde{y}^{m-1}}{p_j^{m-1}} = \prod_{i \neq k} q_k^{m-1} \prod_{j \neq l} p_l^{m-1}, \quad \{q_i, q_j\} = 0, \{p_i, p_j\} = 0. \tag{28}$$

From the explicit form of the brackets it is evident that functions $F_i(u)$ and $F_j(v)$ are indeed the Casimir functions, and the symplectic leaves of this bracket are 2-dimensional. By the Darboux theorem on this symplectic leaves one may, at least locally, find coordinates x and p with the canonical Poisson bracket.

It is easy to see that for the local coordinates on the product of the curves one may take functions u and v. In their terms the corresponding Poisson bracket has the following form:

$$\{u, v\} = m^2 y(u) \widetilde{y}(v).$$

The (local) canonical Darboux coordinates are given by the integrals:

$$x = \frac{1}{m} \int \frac{du}{y^{m-1}(u)}, p = \frac{1}{m} \int \frac{du}{m\widetilde{y}^{m-1}(v)}.$$

Example 5. If we take in the formulas (26)-(27) $m = 2$ we obtain a product of two coverings (see Example 1) of the hyperelliptic curves defined by the equation

$$y^2(u) = \prod_{i=1}^{n}(u - e_i), \quad \widetilde{y}^2(v) = \prod_{i=1}^{n}(v - e_i') \text{ where } q_i^2 \equiv (u - e_i), \quad p_i^2 \equiv (v - e_i').$$

In the terms of local coordinates u, v the corresponding Poisson bracket has the form:

$$\{u, v\} = 4y(u)\widetilde{y}(v).$$

Canonical Darboux coordinates for this bracket are given by the hyperelliptic integrals:

$$x = \int \frac{du}{2y(u)}, p = \int \frac{du}{2\widetilde{y}(v)}.$$

Let us now consider the Hamiltonian equations of motion corresponding to the brackets (28) and to the special choices of the Hamiltonians.

2.5.2 Free motion on the "doubled" higher order curves.

Let us consider the free particle-type polynomial Hamiltonian on the doubled curve:

$$H = \frac{1}{m}\sum_{i=1}^{n} p_i^m. \tag{29}$$

The Hamiltonian equations of motion have the following form:

$$\frac{dq_i}{d\tau} = \widetilde{y}^{m-1}(v)\prod_{i \neq k} q_k^{m-1}, \; \frac{dp_i}{d\tau} = 0. \tag{30}$$

Taking into account that on the level sets of the Hamiltonian we have that $v = const$ we obtain that Hamiltonian equations of motion (30) are equivalent to the generalized Fairlie equations (15).

Example 6. In the case of doubled curves (10) Hamiltonian (29) is the free particle Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 \tag{31}$$

The Hamiltonian equations of motion have the following form:

$$\frac{dq_i}{d\tau} = \widetilde{y}(v)\prod_{i \neq k} q_k, \; \frac{dv}{d\tau} = 0 \tag{32}$$

and coincide on the level sets of Hamiltonian $H = const$ with the "elegant" Fairlie equations.

2.5.3 "Doubled" generalized Fairlie system

Let us consider Hamiltonian system with the following "higher degree oscillator" Hamiltonian:

$$H = \frac{1}{2n}(\sum_{i=1}^{n} p_i^m + \sum_{i=1}^{n} q_i^m). \tag{33}$$

The corresponding equations of motion have the following form:

$$\frac{dq_i}{d\tau} = \prod_{l=1,n} p_l^{m-1} \prod_{i\neq k} q_k^{m-1}, \quad \frac{dp_i}{d\tau} = -\prod_{l=1,n} q_l^{m-1} \prod_{i\neq k} p_k^{m-1} \tag{34}$$

System (38) may be called a "double" of generalized Fairlie integrable system.

In the terms of functions u and v it has the following form:

$$\frac{du}{dt} = m(y(u)\widetilde{y}(v))^{m-1}, \quad \frac{dv}{dt} = -m(y(u)\widetilde{y}(v))^{m-1} \tag{35}$$

This system of equation can be easily integrated:

$$\frac{du}{dt} = m(\widetilde{y}(2E-u)y(u))^{m-1} \Rightarrow \ t = \int \frac{du}{m(y(u)\widetilde{y}(2E-u))^{m-1}}, \tag{36}$$

where we have taken into account that on the trajectories $H = 1/2(v+u) = E$ is constant and have put for simplicity $\sum\limits_{i=1}^{n} e_i = \sum\limits_{i=1}^{n} e_i' = 0$.

Example 7. Let us now put in the above formulas $n = 2$, i.e. let us consider system on the product of two second order curves (10) with the second order Hamiltonian:

$$H = \frac{1}{2n}(\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} q_i^2). \tag{37}$$

This is the Hamiltonian of ordinary oscillator living on the "double-hyperelliptic" curve.

The corresponding equations of motion have the following form:

$$\frac{dq_i}{d\tau} = \prod_{l=1,n} p_l \prod_{i\neq k} q_k, \quad \frac{dp_i}{d\tau} = -\prod_{l=1,n} q_l \prod_{i\neq k} p_k. \tag{38}$$

System (38) may be called a "double" of the Fairlie "elegant" integrable system.

Remark 3. In the case of the rational degeneration $e_i \to 0$, $e_i' \to 0$ and $n > 1$ Hamiltonian system with the Hamiltonian (37) does not go to the usual oscillator because in this case bracket (28) is not canonical. In the case $n = 1$,$m = 2$ bracket (28) is canonical and our system coincide with the usual oscillator.

In the elliptic $n = 3$ case Hamiltonian (37) acquires the form:

$$H = 1/6(\wp(p) + \wp(x)).$$

We may call the system with such a Hamiltonian a "two-particle double-elliptic" Calogero system. That is why systems (37) may be also called "two-particle double hyperelliptic" Calogero system.

Remark 4. Warning: This system is quite different from the 2-particle "double-elliptic" system of [11],[12] which is self-dual with respect to an appropriate "Ruijsenaars duality". Our "free-double elliptic" Calogero system is also a self-dual with respect to the Fourier-Legendre transformation which replace the $x-$torus by the momentum $p-$torus(a sort of the "mirror" transform). This tori are "independent" (they have different moduli) and the initial three-particle motion is replaced by two independent particles which are living each on its own torus. The "double-elliptic" system of [11],[12] moves on a sort of bi-elliptic surface (which is in fact a part of a Kummer surface parameterized by Adler-Van Moerbeke type system of 5 quadrics (see the details in [10]).

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