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PARAMETER-DEPENDENT ASSOCIATIVE YANG-BAXTER EQUATIONS AND POISSON BRACKETS

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We discuss associative analogues of classical Yang-Baxter equation meromorphically dependent on parameters. We discover that such equations enter in a description of a general class of parameter-dependent Poisson structures and double Lie and Poisson structures in sense of M. Van den Bergh. We propose a classification of all solutions for one-dimensional associative Yang-Baxter equations.

Keywords: Poisson brackets; double Poisson structures; Yang-Baxter equations.

1. Introduction

Let $\mathcal{A}$ be an associative unital algebra over $\mathbb{C}$ containing a Lie algebra $\mathfrak{g}$. The parameter-dependent quantum $R-$ matrix $R(u)$ is an invertible $\mathcal{A}\otimes\mathcal{A}-$valued solutions of Quantum Yang-Baxter Equation (QYBE). This equation is a functional relation depending on a complex parameter u

$$R^{12}(u-v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u-v)$$

which plays an important role in the theory of quantum integrable systems. Roughly speaking, any quantum $R-$matrix gives rise to a quantum integrable system.

A classical analogue of the QYBE - the Classical Yang-Baxter Equation (CYBE) can be conventionally written as

$$[r_{12}(u-v), r_{13}(u)+r_{23}(v)] + [r_{13}(u), r_{23}(v)] = 0,$$

where $r(u)$ is in the following relation with $R(u)$:

$$R(u) = \mathbb{I}\otimes\mathbb{I} + \hbar r(u) + O(\hbar).$$

It is clear that $r(u)$ is a $\mathfrak{g}\otimes\mathfrak{g}-$valued function.

The problem of finding solutions to QYBE and CYBE (quantum $R-$matrices and classical $r-$matrices) was one of central and intriguing problems in modern Mathematical Physics during the last forty years. There are many beautiful and explicit results describing the solutions of the YBE's. The first general classification of solutions to the CYBE for complex simple Lie algebras is due to Belavin and Drinfeld [5]. They showed that there exist three different types of the parameter dependence for the classical $r-$matrix: rational, trigonometric and elliptic.

The Inverse Scattering Method provides the existence of intimate relations between the YBE and Hamiltonian (Poisson) structures of the corresponding integrable systems. For example, the classical $r-$matrix induces a Poisson bracket on the corresponding Lie group $G = \mathrm{Lie}(\mathfrak{g})$ such that the group operation is a Poisson morphism (a Poisson-Lie group structure of V. Drinfeld).

A third version of YBE: a (parameter-dependent) Associative Yang-Baxter Equation (AYBE) (see the definition below) had appeared independently in attempts to understand a nature of the associativity constraint or "Massey triple relations" in the framework of the homology mirror symmetry on elliptic curves (A. Polishchuk, 1999-2000, [10]) and (as a constant or parameter independent) counterpart of an AYBE - in a description of some associative version of the Lie algebra Drinfeld double (M. Aguiar, 2000, [1]). We remark also that the similar constant AYBE emerges in the paper of S. Fomin and A. Kirillov . They have introduced a model for the cohomology ring of the flag manifold $X = \mathrm{SL}_n/B$, using a non-commutative algebra defined by the quadratic relations satisfied by the divided difference operators corresponding to the positive roots. (1995-96 [7]). One of the relations in their algebra coincides with those in [1]. We should note also that a 1-dimensional multiplicative AYBE analogue of [10] was discovered by B. Feigin and one of the authors (A.O.) as the exchange relations for elliptic Sklyanin algebras (1993, [4]).

An interesting relations of CYBE and AYBE with vector bundle geometry on singular and degenerated cubic curves were described in papers of I. Burban with collaborators [12]. They extend the ideas and results of A. Polishschuk to much wider class of plane cubic curves and their degenerations. They find in this frame many solutions of the matrix-valued AYBEs and CYBEs appeared within their algebro-geometric picture.

One should remark that the solutions of AYBE equations sometimes are related to QYBE and CYBE but are essentially different. Solving the AYBE we solve often in the same time a CYBE and QYBE but not vice-versa.

As we have observed above a motivation to study various YBE comes from the Integrable Systems. A natural question of an associative algebra analogue for the Hamiltonian formalism and the related Poisson structures arises in the theory of integrable systems with matrix variables. An intuitive version of it was defined by one of the authors (V.S.) with various collaborators in [11], [6]. More systematic approach was developed in [15]. In this paper we studied special Poisson structures on the representations of free associative algebras, following the Kontsevich "Representation functor" philosophy [13].

This class of *trace* Poisson brackets is closely related [18] to the Van den Bergh *double Poisson structures* (see [14]). Any double Poisson structure on an associative algebra $\mathcal{A}$ defines a Poisson structure on the trace space $\mathcal{A}/[\mathcal{A},\mathcal{A}]$. The Poisson brackets considered in [15] correspond the case when $\mathcal{A}$ is the free associative algebra. In [18] we have studied (and classified for "low-dimensional" case) *quadratic* double Poisson structures on free associative algebras and clarified their connection with the constant AYBE. In this context our results are complementary to those in [17].

The goal of this paper is to discuss generalizations of the above concepts and relations between them to the parameter-dependent case. We establish connections between the (parameter-dependent) AYBE introduced by A. Polishchuk in [10] and several notions of mathematical physics such that parameter dependent intertwining quadratic algebras, linear and quadratic Poisson brackets and their double Poisson analogues.

We propose also a full classification of the parameter-dependent AYBE solutions (which à priori don't depend on differences of arguments) for the case of $m = 1$.) Some of solutions can be easily generalized to higher dimension cases and coincide with solutions obtained by A. Polishchuk and by I. Burban with co-authors.

The volume limit does not permit us to consider many other aspects of the parameter-dependent Poisson and double Poisson structures and their connections with functional associative algebras. In particular, the bi-Hamiltonian properties and constraints and the corresponding examples of their solutions are beyond the scope of the paper. Another interesting open question is to study "higher" (cubic, quartic etc.) parameter-dependent Poisson structures. The evidence to their existence is given by their examples in the "constant" case discovered in [6]. An intriguing relations between the Sklyanin elliptic algebras, their intertwinners and many-parametric solutions of the AYBE deserve deep studies. We hope to come back to these questions in other publications.

2. Associative Yang-Baxter equation

2.1. *Constant AYBE*

By definition, a classical r-matrix on an associative $\mathbb{C}-$ algebra $\mathcal{A}$ is any solution of the following tensor equations

$$r^{12} = -r^{21} \tag{2.1}$$

and

$$r^{12}r^{23} = r^{13}r^{12} + r^{23}r^{13}. \tag{2.2}$$

Here $r \in \mathcal{A} \otimes \mathcal{A}$ and (2.2) is a relation defined in the tensor cube $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. Relation (2.2) is called *a classical associative Yang-Baxter equation.*

Rewrite (2.2) as $A(r) := r^{12}r^{23} - r^{13}r^{12} - r^{23}r^{13} = 0$, apply the "flip" operator P^{13} (the transposition on the first and third factors) to (2.2) and use the skew-symmetry (2.1). We obtain the "conjugated" AYBE $A^*(r) = r^{23}r^{12} - r^{13}r^{23} - r^{12}r^{13} = 0$. The difference $[[r]] := A(r) - A^*(r)$ is a skew-symmetric element in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. The equation

$$[[r]] := [r^{12}, r^{13}] + [r^{12}r^{23}] + [r^{13}r^{23}] = 0. \tag{2.3}$$

is nothing but the *constant classical Yang-Baxter equation.*

In the case when $\mathcal{A} = \mathrm{Mat_m}(\mathbb{C})$ the equations (2.1), (2.2) are equivalent to

$$r^{\lambda\sigma}_{\alpha\beta}r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau}r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha}r^{\lambda\mu}_{\sigma\beta} = 0, \qquad r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha}. \tag{2.4}$$

Here $r = r^{km}_{ij} e^i_k \otimes e^j_m$, where e^i_j are the matrix unities: $e^j_i e^m_k = \delta^j_k e^m_i$.

The tensor r may be also interpreted

1) as an operator on $\mathbb{C}^m \otimes \mathbb{C}^m$;

2) as an operator on $\mathrm{Mat}_m(\mathbb{C})$.

For the first interpretation all operators act in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, σ^{ij} means the transposition of i-th and j-th components of the tensor product, and r^{ij} means the operator r acting in the product of the i-th and j-th components.

For the second interpretation, we define operators $r, \bar{r} : Mat_m(\mathbb{C}) \to Mat_m(\mathbb{C})$ by $r(x)^p_q = r^{mp}_{nq} x^n_m$, $\bar{r}(x)^p_q = r^{pm}_{nq} x^n_m$. Then (2.1), (2.2) provide the following operator identities:

$$r(x) = -\bar{r}(x), \qquad r(x)r(y) = r(xr(y)) + r(x)y),$$
$$\bar{r}(x)\bar{r}(y) = \bar{r}(x\bar{r}(y)) + \bar{r}(x)y),$$

for any x, y. These identities mean that operators r and $\bar{r}$ satisfies the *Rota-Baxter equation* [16] and this fact implies also that the new matrix multiplications $\circ_r$ and $\circ_{\bar{r}}$ defined by

$$x \circ_r y = r(x)y + xr(y), \qquad x \circ_{\bar{r}} y = \bar{r}(x)y + x\bar{r}(y) \tag{2.5}$$

are associative.

2.2. *General AYBE with four parameters*

The most interesting applications of classical r-matrices related to Lie algebras are referred to the parameter-dependent case. In general, r depends on two parameters u, v but usually one assumes that $r = r(u - v)$. Remarkably, the most general associative r-matrix depends on four parameters!

Definition. (cf. the matrix algebra case in [10] and [12]) A classical parameter dependent r-matrix on an associative algebra $\mathcal{A}$ is any solution of the following functional equations

$$r^{12}(u, v, x, y) = -r^{21}(v, u, y, x) \tag{2.6}$$

and

$$r^{12}(u, v, x, y)r^{23}(u, w, y, z) = r^{13}(u, w, x, z)r^{12}(w, v, x, y)+r^{23}(v, w, y, z)r^{13}(u, v, x, z). \tag{2.7}$$

Here $r : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathcal{A} \otimes \mathcal{A}$ is a meromorphic $\mathcal{A} \otimes \mathcal{A}-$ valued function and (2.7) is a relation defined in the tensor cube $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

This definition is the straightforward generalization of the four parameter-dependent unitary AYBE which was defined in [10] and extensively studied in [12]. Let us remind the context of the A. Polischuk's definition. Let $\mathcal{E}$ be a smooth (for simplicity) projective elliptic curve and $E \in \mathrm{Vect}(\mathcal{E})$ will be always a simple ($\mathrm{End}(E) = \mathbb{C}$) vector bundle on it. The fibre of E in a point $x \in \mathcal{E}$ will be denote by E_x. Let $E_1, E_2 \in Vect(\mathcal{E})$ be two simple vector bundles such that $\mathrm{Ext(E_1, E_2)} = 0$. Then for distinct points $x_1, x_2 \in \mathcal{E}$ one can define a tensor

$$r^{E_1,E_2}_{x_1,x_2} \in \mathrm{Hom}(E^*_{1,x_1}, E^*_{2,x_1}) \otimes \mathrm{Hom}(E^*_{2,x_2}, E^*_{1,x_2}) \tag{2.8}$$

such that for a triple of simple vector bundles $E_1, E_2, E_3 \in \mathrm{Vect}(\mathcal{E})$ pairwise satisfying the above conditions and for a triple of distinct points $x_1, x_2, x_3 \in \mathcal{E}$ the following relations in $\mathrm{Hom}(E^*_{1,x_1}, E^*_{2,x_1}) \otimes \mathrm{Hom}(E^*_{2,x_2}, E^*_{3,x_2}) \otimes \mathrm{Hom}(E^*_{3,x_3}, E^*_{1,x_3})$ satisfies:

$$(r^{E_1,E_2}_{x_1,x_2})^{21} = -r^{E_2,E_1}_{x_2,x_1} \tag{2.9}$$

and

$$(r^{E_3,E_2}_{x_1,x_2})^{12}(r^{E_1,E_3}_{x_1,x_3})^{13} - (r^{E_1,E_3}_{x_2,x_3})^{23}(r^{E_1,E_2}_{x_1,x_2})^{12} + (r^{E_1,E_2}_{x_1,x_3})^{13}(r^{E_2,E_3}_{x_2,x_3})^{23} = 0 \tag{2.10}$$

(see [10].) This tensor expresses a certain triple Massey product in the derived category of $\mathcal{E}$ and the relation (2.10) is a consequence of the associativity constraint corresponding to the $A_\infty-$ structure related to this category. To get the parameter-dependent AYBE one should uniformize the curve $\mathcal{E}$, trivialize all vector bundles and express the tensor (2.8) as a $\mathrm{Mat}_m(\mathbb{C}) \otimes \mathrm{Mat}_m(\mathbb{C})-$ valued function of complex variables $r_{12}(u, v, x, y)$ where (u, v) correspond to E_1, E_2 and $(x, y)-$ to x_1, x_2 (see [10]). It should be stressed that the AYBE (2.10) gives the AYBE depending on the differences of the variables and the $r-$matrix $r_{12}(u, v, x, y) = r_{12}(u - v, x - y)$.

3. Parameter-dependent AYBE and related Poisson-like structures

3.1. *Quadratic algebra with two parameters.*

A version of the AYBE with four parametres has appeared at the first time in the paper of B. Feigin and one of the authors (A.O.) [4]. They considered a quadratic algebra with generators $X(u,x)$ and commutation relations

$$X(u,x)X(v,y) = \beta(u,v,x,y)X(v,y)X(u,x) + \alpha(u,v,x,y)X(u,y)X(v,x) \quad (3.11)$$

has been considered. A priori it was assumed that the coefficients α and β only depend on the differences $u-v$ and $x-y$. In what follows we precise the results from Ch.3 of ([4], p. 43-44) and remove the differences dependence for both parameter pairs.

Applying the quadratic commutation rule (3.11) to the products in r.h.s. of (3.11), we obtain that

$$\beta(u,v,x,y)\beta(v,u,y,x) + \alpha(u,v,x,y)\alpha(u,v,y,x) = 1 \quad (3.12)$$

and

$$\beta(u,v,x,y)\alpha(v,u,y,x) + \alpha(u,v,x,y)\beta(u,v,y,x) = 0. \quad (3.13)$$

Reducing the product $X(u,x)X(v,y)X(w,z)$ to $X(w,z)X(v,y)X(u,x)$ by two different ways, we get quantum triangle Yang-Baxter relations. They are equivalent to five functional equations for α and β. It follows from two of them that if $\alpha \neq 0$ then β is a product of the form

$$\beta = f_1(u,v)f_2(u,y)f_3(v,x)f_4(x,y).$$

Consider a special case $f_2 = f_3 = 1$ [a]. Let

$$\beta(u,v,x,y) = \frac{\psi(u,v)}{\phi(x,y)}, \qquad \alpha(u,v,x,y) = \frac{r(u,v,x,y)}{\phi(x,y)}.$$

Then (3.13) implies

$$r(u,v,x,y) = -r(v,u,y,x), \quad (3.14)$$

and (3.12) gives rise to

$$r(u,v,x,y)r(u,v,y,x) = \phi(x,y)\phi(y,x) - \psi(u,v)\psi(v,u).$$

Two of the Yang-Baxter relations turn out to be equivalent to

$$r(w,u,z,x)r(v,w,y,x) + r(u,v,x,y)r(w,u,z,y) + r(v,w,y,z)r(u,v,x,z) = 0. \quad (3.15)$$

The last Yang-Baxter relation gives

$$(\psi(v,w)\psi(w,v) - \psi(u,v)\psi(v,u))\, r(u,w,x,z) =$$

[a] This special case is necessarily appeared when the functions β depend on differences of the arguments. We postpone the study of more general case to future works for the paper volume limits reasons

$$r(u,v,x,y)r(u,v,y,z)r(v,w,x,z) - r(u,v,x,z)r(v,w,x,y)r(v,w,y,z).$$

Equations (3.14), (3.15) coincide (for $A = \mathbb{C}$) with (2.6), (2.7). Solutions of these equations are discussed in Section 4. It turns out that the corresponding functions ψ and ϕ exist for any solution of (3.14), (3.15). In general case we have an analog of equations (3.14), (3.15), which involve the functions f_2, f_3.

3.2. *Quadratic Poisson brackets.*

We remind the well-known conditions for quadratic *local* Poisson algebra with generators $e_i(u)$, where $i = 1, ..., m$ depending on $u \in \mathbb{C}$. The locality means that

$$\{e_i(u), e_j(v)\} = P_{ij},$$

where P_{ij} are some polynomials in $e_1(u), e_1(v), ..., e_m(u), e_m(v)$. As usual in the most interesting examples of parameter dependent Poisson brackets, the coefficients of this polynomials may have the pole singularities on the divisor $u = v$.

The condition of "quadraticity" implies the following bracket form:

$$\{e_i(u), e_j(v)\} = \alpha_{ij}^{ml}(u,v)e_m(u)e_l(v),$$

where

$$\alpha_{ij}^{ml}(u,v) = -\alpha_{ji}^{lm}(v,u)$$

from the skew-symmetry constraint for the Poisson structure.

The Jacobi identity gives the following 6-term relation:

$$\begin{aligned} &\alpha_{ij}^{\sigma l}(u,v)\alpha_{k\sigma}^{rs}(w,u) + \alpha_{jk}^{\sigma r}(v,w)\alpha_{i\sigma}^{sl}(u,v) + \alpha_{ki}^{\sigma s}(w,u)\alpha_{j\sigma}^{lr}(v,w) - \\ -&\alpha_{ji}^{\sigma s}(v,u)\alpha_{k\sigma}^{rl}(w,v) - \alpha_{kj}^{\sigma l}(w,v)\alpha_{i\sigma}^{sr}(u,w) - \alpha_{ik}^{\sigma r}(u,w)\alpha_{j\sigma}^{ls}(v,u) = 0. \end{aligned} \tag{3.16}$$

The equation (3.16) is the classical Yang-Baxter equation (CYBE) which is a "difference" of two AYBE:

$$[[\alpha, \alpha]] = A(\alpha) - A^*(\alpha),$$

where $A(\alpha)$ is the parameter dependent analogue of (2.2) while $A^*(\alpha)$ is its "conjugation".

3.2.1. *Two-parametric quadratic Poisson brackets*

It is not difficult to generalize the above conditions to the case of generators $e_i(x,u)$ meromorphically depending on two complex parameters (u,x). Our first observation is that the *same* associative Yang-Baxter equation (2.7) appears in the two-parameter Poisson conditions. Consider the intertwining operation of the form

$$\{e_i(u,x), e_j(v,y)\} = \beta_{ij}^{ml}(u,v,x,y)e_m(u,x)e_l(v,y) + r_{ij}^{ml}(u,v,x,y)e_m(u,y)e_l(v,x), \tag{3.17}$$

where $i, j = 1, ..., m$.

Theorem 1. *The operation for (3.17) satisfies all axioms of a Poisson bracket iff*

$$r^{ml}_{ij}(u,v,x,y) = -r^{lm}_{ji}(v,u,y,x), \tag{3.18}$$

and

$$\beta^{ml}_{ij}(u,v,x,y) = -\beta^{lm}_{ji}(v,u,y,x).$$

and

$$r^{\sigma q}_{ki}(w,u,z,x)r^{rs}_{j\sigma}(v,w,y,x)+r^{\sigma r}_{ij}(u,v,x,y)r^{sq}_{k\sigma}(w,u,z,y)+r^{\sigma s}_{jk}(v,w,y,z)r^{qr}_{i\sigma}(u,v,x,z) = 0, \tag{3.19}$$

with

$$\begin{aligned} &r^{\sigma q}_{ij}(u,v,x,y)\beta^{rs}_{k\sigma}(w,u,z,y) - r^{\sigma s}_{ji}(v,u,y,x)\beta^{rq}_{k\sigma}(w,v,z,x) + \\ &\beta^{\sigma r}_{jk}(v,w,y,z)r^{sq}_{i\sigma}(u,v,x,y) - \beta^{\sigma r}_{ik}(u,w,x,z)r^{qs}_{j\sigma}(v,u,y,x) = 0, \end{aligned} \tag{3.20}$$

and with

$$\begin{aligned} &\beta^{\sigma s}_{jk}(v,w,y,z)\beta^{qr}_{i\sigma}(u,v,x,z) - \beta^{\sigma q}_{ji}(v,u,z,x)\beta^{sr}_{k\sigma}(w,v,z,y) + \\ &\beta^{\sigma q}_{ki}(w,u,z,x)\beta^{rs}_{j\sigma}(v,w,y,x) - \beta^{\sigma r}_{kj}(w,v,x,y)\beta^{qs}_{i\sigma}(u,w,x,z) + \\ &\beta^{\sigma r}_{ij}(u,v,x,y)\beta^{sq}_{k\sigma}(w,u,z,y) - \beta^{\sigma s}_{ik}(u,w,y,z)\beta^{rq}_{j\sigma}(v,u,y,x) = 0. \end{aligned} \tag{3.21}$$

are satisfied.

If we denote

$$R(u,v,x,y) = r^{jm}_{ik}(u,v,x,y)e^i_j \otimes e^k_m,$$

where e^i_j are matrix unities in $\mathrm{Mat}_m(\mathbb{C})$, then equations (3.19) are nothing but the associative Yang-Baxter equation (2.7) for $\mathcal{A} = \mathrm{Mat}_m(\mathbb{C})$.

The equation (3.21) is again, like in the formulas (3.16) for two-parameter case, a four-parameter-dependent analogue of the classical Yang-Baxter equation (CYBE) (2.3) which is a difference of two AYBE:

$$[[B,B]] = A(B) - A^*(B),$$

where $B(u,v,x,y) = \beta^{ml}_{ij}(u,v,x,y)e^i_m \otimes e^j_l$ and $A(B)$ is exactly the "positive sign" part of the equation (3.21) while $A^*(B)$ is its "conjugation" which coincides with the "negative sign" part of the same equation.

In the case $m = 1$ the equation (3.19) coincides with (3.15), the equation (3.21) are satisfied identically and the equation (3.20) is reduced to

$$B(w,u,z,y) - B(w,u,z,x) + B(v,w,y,z) - B(v,w,x,z) = 0.$$

The general skew-symmetric solution of this equation is given by

$$B(u,v,x,y) = a(u,v) + b(x,y) + c(v,x) - c(u,y),$$

where $a(u,v) = -a(v,u)$, $b(x,y) = -b(y,x)$.

3.3. *Double Poisson brackets with parameters*

Our second observation is that relations (3.18), (3.19) describe a class of double Poisson brackets (see [14]) with two parameters. Let us remind the basic definition.

Definition 1. *(M. Van den Bergh). A double Poisson bracket on an associative algebra $\mathcal{A}$ is a $\mathbb{C}$-linear map $\{\!\{,\}\!\} : \mathcal{A}\otimes\mathcal{A} \mapsto \mathcal{A}\otimes\mathcal{A}$ satisfying the following conditions:*

$$\{\!\{u, v\}\!\} = -\{\!\{v, u\}\!\}^{\circ}, \tag{3.22}$$

$$\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l + \sigma\{\!\{v, \{\!\{w, u\}\!\}\}\!\}_l + \sigma^2\{\!\{w, \{\!\{u, v\}\!\}\}\!\}_l = 0, \tag{3.23}$$

and

$$\{\!\{u, vw\}\!\} = (v \otimes 1)\{\!\{u, w\}\!\} + \{\!\{u, v\}\!\}(1 \otimes w). \tag{3.24}$$

Here $(u \otimes v)^{\circ} := v \otimes u$; $\{\!\{v_1, v_2 \otimes v_3\}\!\}_l := \{\!\{v_1, v_2\}\!\} \otimes v_3$ and $\sigma(v_1 \otimes v_2 \otimes v_3) := v_3 \otimes v_1 \otimes v_2$.

If the bracket $\{\!\{,\}\!\} : \mathcal{A}\otimes\mathcal{A} \mapsto \mathcal{A}\otimes\mathcal{A}$ satisfies only to (3.22) and (3.23) one can say that this is just a *double Lie bracket*.

The main property of double Poisson bracket for us is the following relation between double and usual Poisson brackets established by M.Van den Bergh [14]. Let μ denote the multiplication map $\mu : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ i.e. $\mu(u \otimes v) = uv$. We define a $\mathbb{C}-$bilinear bracket operation in $\mathcal{A}$ by $\{-,-\} := \mu(\{\!\{-,-\}\!\})$.

Proposition 1. *Let $\{\!\{-,-\}\!\}$ be a double Poisson bracket on $\mathcal{A}$. Then $\{-,-\}$ is an $H-$ Poisson bracket of W. Crawley-Boevey on $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ [9] which is defined as*

$$\{\bar{a}, \bar{b}\} = \overline{\mu(\{\!\{a, b\}\!\}}, \tag{3.25}$$

where $\bar{a}$ means the image of $a \in \mathcal{A}$ under the natural projection $\mathcal{A} \to \mathcal{A}/[\mathcal{A},\mathcal{A}]$.

Remark 1. $H-$ Poisson bracket of W. Crawley-Boevey (strictly speaking) is not a Poisson bracket (the quotient of "traces" $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ is not an algebra) but it satisfies the following important property: for any $b \in A$ the endomorphism $[\bar{b},]$ of $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ is induced by some derivative ∂_b of $\mathcal{A}$:

$$[\bar{b}, \bar{c}] = \overline{\partial_b(c)}, \quad c \in \mathcal{A}.$$

We shall consider in what follows the "double analogues" of local parameter dependent Poisson brackets with two complex parameters. Define a double bracket operation given on generators $\mathbf{E}(u, x)$, where $u, x \in \mathbb{C}$, by the formula

$$\{\!\{\mathbf{E}_i(u, x), \mathbf{E}_j(v, y)\}\!\} =$$

$$r_{ij}^{km}(u, v, x, y)\mathbf{E}_k(u, y) \otimes \mathbf{E}_m(v, x) + \beta_{ij}^{km}(u, v, x, y)\mathbf{E}_k(v, x) \otimes \mathbf{E}_m(u, y)+ \tag{3.26}$$

$$\gamma_{ij}^{km}(u, v, x, y)\mathbf{E}_k(u, x) \otimes \mathbf{E}_m(v, y) + \delta_{ij}^{km}(u, v, x, y)\mathbf{E}_k(v, y) \otimes \mathbf{E}_m(u, x).$$

Theorem 2. *The operation (3.26) gives an example of (parameter-dependent) double Lie algebra iff*

$$r_{ij}^{km}(u,v,x,y) = -r_{ji}^{mk}(v,u,y,x), \qquad \beta_{ij}^{km}(u,v,x,y) = -\beta_{ji}^{mk}(v,u,y,x),$$

$$\gamma_{ij}^{km}(u,v,x,y) = -\gamma_{ji}^{mk}(v,u,y,x), \qquad \delta_{ij}^{km}(u,v,x,y) = -\delta_{ji}^{mk}(v,u,y,x)$$

and the following chain of relations satisfies:

$$\begin{gathered} r_{jk}^{\sigma m}(v,w,y,z) r_{i\sigma}^{pq}(u,v,x,z) + r_{ki}^{\sigma p}(w,u,z,x) r_{j\sigma}^{qm}(v,w,y,x) + \\ + r_{ij}^{\sigma q}(u,v,x,y) r_{k\sigma}^{mp}(w,u,z,y) = 0; \end{gathered} \tag{3.27}$$

$$\begin{gathered} \beta_{jk}^{\sigma m}(v,w,y,z) \beta_{i\sigma}^{pq}(u,w,x,y) + \beta_{ki}^{\sigma p}(w,u,z,x) \beta_{j\sigma}^{qm}(v,u,y,z) + \\ + \beta_{ij}^{\sigma q}(u,v,x,y) \beta_{k\sigma}^{mp}(w,v,z,x) = 0; \end{gathered} \tag{3.28}$$

$$r_{jk}^{\sigma m}(v,w,y,z) \beta_{i\sigma}^{pq}(u,v,x,z) = 0; \tag{3.29}$$

$$\beta_{jk}^{\sigma m}(v,w,y,z) r_{i\sigma}^{pq}(u,w,x,y) = 0; \tag{3.30}$$

$$\begin{gathered} \delta_{jk}^{\sigma m}(v,w,y,z) \delta_{i\sigma}^{pq}(u,v,x,z) + \delta_{ki}^{\sigma p}(w,u,z,x) \delta_{j\sigma}^{qm}(v,w,y,x) + \\ + \delta_{ij}^{\sigma q}(u,v,x,y) \delta_{k\sigma}^{mp}(w,u,z,y) = 0; \end{gathered} \tag{3.31}$$

$$\begin{gathered} \gamma_{jk}^{\sigma m}(v,w,y,z) \gamma_{i\sigma}^{pq}(u,w,x,y) + \gamma_{ki}^{\sigma p}(w,u,z,x) \gamma_{j\sigma}^{qm}(v,u,y,z) + \\ + \gamma_{ij}^{\sigma q}(u,v,x,y) \gamma_{k\sigma}^{mp}(w,v,z,x) = 0; \end{gathered} \tag{3.32}$$

$$r_{jk}^{\sigma m}(v,w,y,z) \gamma_{i\sigma}^{pq}(u,v,x,z) + \gamma_{ki}^{\sigma p}(w,u,z,x) r_{j\sigma}^{qm}(v,w,y,z) = 0; \tag{3.33}$$

$$\beta_{jk}^{\sigma m}(v,w,y,z) \gamma_{i\sigma}^{pq}(u,w,x,y) + \gamma_{ki}^{\sigma p}(w,u,z,x) \beta_{j\sigma}^{qm}(v,w,y,z) = 0; \tag{3.34}$$

$$r_{jk}^{\sigma m}(v,w,y,z) \delta_{i\sigma}^{pq}(u,v,x,z) + \delta_{ij}^{\sigma q}(u,v,x,y) r_{k\sigma}^{mp}(w,v,z,y) = 0; \tag{3.35}$$

$$\beta_{jk}^{\sigma m}(v,w,y,z) \delta_{i\sigma}^{pq}(u,w,x,y) + \delta_{ij}^{\sigma q}(u,v,x,y) \beta_{k\sigma}^{mp}(w,v,z,y) = 0; \tag{3.36}$$

$$\gamma_{jk}^{\sigma m}(v,w,y,z) \delta_{i\sigma}^{pq}(u,v,x,y) + \delta_{ij}^{\sigma q}(u,v,x,y) \gamma_{k\sigma}^{mp}(w,v,z,y) = 0. \tag{3.37}$$

The first four relations (a skew-symmetry) are resulted from (3.22) and the Jacobi identity (3.23) implies the other functional relations

Remark 2. We remark that there are two conjugated AYBE $A(r) = 0$ (3.27) and $A^*(\beta) = 0$ (3.28). The equation (3.31) is the same AYBE $A(\delta) = 0$ as in (3.27) and the equation (3.32) is the AYBE $A^*(\gamma) = 0$ for (3.28).

Remark 3. If $\gamma = \delta = 0$ the double brackets (3.26) are called the (purely) exchange brackets.

Remark 4. If we extend this double Lie structure by the Leibniz rule of Van den Bergh (3.24) we obtain a parameter dependent double Poisson structure on the functional algebra $\mathcal{A}$. In what follows we shall always suppose such extension when we speak about various examples of double Poisson structures.

Remark 5. It is worth to observe that if we apply the construction described in Prop.1 to the double brackets 3.26 then we obtain the quadratic Poisson parameter-dependent brackets (3.17) as related trace-Poisson brackets on the Rep($\mathcal{A}$). The relations (3.27)-(3.37) imply the conditions (3.19)-(3.21) for corresponding brackets in (3.17) with the identifications $\delta + \gamma \equiv B$ and $r + \beta \equiv R$.

3.3.1. *Quadratic one-parameter double brackets*

Consider the quadratic double Poisson brackets in the case when the generators depend on only one parameter. The skew-symmetric ansatz for such brackets, more general [b] then (3.26), has the form

$$\begin{aligned}\{\!\{\mathbf{E}_i(u), \mathbf{E}_j(v)\}\!\} = r_{ij}^{pq}(u,v)\mathbf{E}_p(u) \otimes \mathbf{E}_q(v) + \beta_{ij}^{pq}(u,v)\mathbf{E}_p(v) \otimes \mathbf{E}_q(u) +\\ \gamma_{ij}^{pq}(u,v)\mathbf{E}_p(u)\mathbf{E}_q(v) \otimes 1 + \delta_{ij}^{pq}(u,v)1 \otimes \mathbf{E}_p(u)\mathbf{E}_q(v) -\\ \delta_{ji}^{pq}(v,u)\mathbf{E}_p(v)\mathbf{E}_q(u) \otimes 1 - \gamma_{ji}^{pq}(v,u)1 \otimes \mathbf{E}_p(v)\mathbf{E}_q(u),\end{aligned} \tag{3.38}$$

where

$$r_{ki}^{pq}(u,v) = -r_{ik}^{qp}(v,u), \qquad \beta_{ki}^{pq}(u,v) = -\beta_{ik}^{qp}(v,u).$$

If the tensors γ and δ vanish then (3.38) looks like a special case of (3.26).

We would like to stress however that the conditions for (3.38) to be a one-parameter double Poisson bracket is weaker then the two-parameter conditions for (3.26). Indeed, it follows from (3.23) that r and β should satisfy equations

$$r_{ki}^{\sigma q}(w,u)r_{j\sigma}^{rs}(v,w) + r_{ij}^{\sigma r}(u,v)r_{k\sigma}^{sq}(w,u) + r_{jk}^{\sigma s}(v,w)r_{i\sigma}^{qr}(u,v) = 0; \tag{3.39}$$

$$\beta_{ki}^{\sigma q}(w,u)\beta_{j\sigma}^{rs}(v,u) + \beta_{ij}^{\sigma r}(u,v)\beta_{k\sigma}^{sq}(w,v) + \beta_{jk}^{\sigma s}(v,w)\beta_{i\sigma}^{qr}(u,w) = 0; \tag{3.40}$$

$$r_{jk}^{\sigma q}(v,w)\beta_{i\sigma}^{rs}(u,v) + \beta_{ij}^{\sigma s}(u,v)r_{k\sigma}^{qr}(w,v) = 0. \tag{3.41}$$

The conditions (3.39) and (3.40) can be obtained from (3.27) and (3.28) by "forgetting" the dependence on the second pair of parameters. Again, one can straightforwardly check that the AYBE of (3.40) is conjugated to the AYBE of (3.39) in a full correspondence with the Remark 3.

[b]A similar two-parametric ansatz in the previous subsection we don't consider only due to volume restrictions.

It is easy to see that the equation (3.39) describes solutions of (2.7) that do not depend on two first arguments whereas solutions of the equation (3.40) correspond to solutions of (2.7) that do not depend on third and fourth arguments.

The condition (3.41) differs from (3.29) and (3.30).

The other relations between the coefficients in (3.38) read as follows:

$$\delta^{pq}_{kj}(w,v)\delta^{rs}_{ip}(u,w) = \delta^{rp}_{ik}(u,w)\delta^{sq}_{pj}(w,v);$$

$$\gamma^{pq}_{jk}(v,w)\gamma^{rs}_{pi}(v,u) = 0, \qquad \gamma^{qp}_{ki}(w,u)\gamma^{rs}_{jp}(v,u) = 0;$$

$$\gamma^{qp}_{ij}(u,v)\gamma^{rs}_{kp}(v,w) = 0, \qquad \gamma^{pq}_{jk}(v,w)\gamma^{rs}_{ip}(u,v) = 0;$$

$$\delta^{qp}_{ji}(v,u)\delta^{rs}_{kp}(w,u) + \delta^{rs}_{pi}(w,u)\beta^{pq}_{jk}(v,w) + \delta^{ps}_{ji}(v,u)\beta^{qr}_{kp}(w,v) = 0;$$

$$\delta^{pq}_{ji}(v,u)\delta^{rs}_{pk}(v,w) + \delta^{rs}_{jp}(v,w)r^{pq}_{ki}(w,u) + \delta^{rp}_{ji}(v,u)r^{qs}_{pk}(u,w) = 0;$$

$$\gamma^{rs}_{ip}(u,v)r^{pq}_{jk}(v,w)+\gamma^{ps}_{ij}(u,v)r^{qr}_{kp}(w,u) = \gamma^{rp}_{ki}(w,u)\beta^{sq}_{jp}(v,u)+\gamma^{rs}_{pi}(w,u)\beta^{qp}_{kj}(w,v) = 0;$$

$$\delta^{pq}_{ik}(u,w)r^{rs}_{jp}(v,u)+\delta^{sq}_{pk}(u,w)r^{pr}_{ij}(u,v) = \delta^{rs}_{ip}(u,w)\beta^{pq}_{jk}(v,w)+\delta^{rp}_{ik}(u,w)\beta^{qs}_{pj}(w,v) = 0;$$

$$\gamma^{qr}_{pk}(v,w)\delta^{ps}_{ji}(v,u) - \gamma^{qp}_{jk}(v,w)\delta^{rs}_{pi}(w,u) = \gamma^{rs}_{ip}(u,v)\delta^{qp}_{kj}(w,v) - \gamma^{pr}_{ij}(u,v)\delta^{qr}_{kp}(v,w) = 0;$$

$$\gamma^{qp}_{ij}(u,v)r^{sr}_{pk}(v,w) + \gamma^{ps}_{ij}(u,v)\delta^{qr}_{pk}(u,v) = \gamma^{rs}_{pj}(w,v)r^{pq}_{ki}(w,u) + \gamma^{rs}_{kp}(w,v)\delta^{pq}_{ji}(v,u) = 0;$$

$$\gamma^{pq}_{ij}(u,v)\beta^{rs}_{kp}(w,u)+\gamma^{qp}_{ij}(u,v)\delta^{rs}_{kp}(w,v) = \gamma^{rs}_{ip}(u,w)\beta^{pq}_{jk}(v,w)+\gamma^{rs}_{pk}(u,w)\delta^{qp}_{ji}(v,u) = 0.$$

In the case $m = 1$ we get $\gamma(u,v) = 0$, and

$$\begin{aligned} &r(v,w)r(u,v) + r(w,u)r(v,w) + r(u,v)r(w,u) = 0, \\ &\beta(v,w)\beta(u,v) + \beta(w,u)\beta(v,w) + \beta(u,v)\beta(w,u) = 0, \\ &\delta(v,u)\delta(v,w) = r(w,u)\Big(\delta(v,u) - \delta(v,w)\Big), \\ &\delta(w,v)\delta(u,v) = \beta(w,u)\Big(\delta(w,v) - \delta(u,v)\Big). \end{aligned} \tag{3.42}$$

The general solution is given by

$$r(u,v) = \frac{1}{g(u)-g(v)}, \quad \beta(u,v) = \frac{1}{h(u)-h(v)}, \quad \delta(u,v) = \frac{\epsilon}{g(v)-h(u)}. \tag{3.43}$$

Here g and h are arbitrary functions of one variable and the constant ϵ equals 1 or 0.

In the case of arbitrary m there exist particular solutions of the form

$$r^{ij}_{pq}(u,v) = \frac{k^{ij}_{pq}}{u-v}, \qquad \beta^{ij}_{pq}(u,v) = -r^{ij}_{pq}(u,v), \qquad \gamma^{ij}_{pq}(u,v) = \delta^{ij}_{pq}(u,v) = 0.$$

where

$$k^{ij}_{pq} = \delta^i_p \delta^j_q \tag{3.44}$$

or

$$k^{ij}_{pq} = \delta^i_q \delta^j_p. \tag{3.45}$$

The corresponding double brackets are given by

$$\{\!\{\mathbf{E}_i(u), \mathbf{E}_j(v)\}\!\} = \frac{\mathbf{E}_i(u) \otimes \mathbf{E}_j(v) - \mathbf{E}_i(v) \otimes \mathbf{E}_j(u)}{u - v}$$

and

$$\{\!\{\mathbf{E}_i(u), \mathbf{E}_j(v)\}\!\} = \frac{\mathbf{E}_j(u) \otimes \mathbf{E}_i(v) - \mathbf{E}_j(v) \otimes \mathbf{E}_i(u)}{u - v}.$$

Solutions (3.44), (3.45) are analogs of the Yang solutions in the theory of classical Yang-Baxter equation on Lie algebras [2]. It is known that in the Lie case the sum of the Yang solution and any constant solution is a solution of the same equation. In our situation it is easy to prove the following

Proposition 2 (cf. [3]). Let $\tilde{r}$ be any solution of (2.4). Then

$$r^{ij}_{pq}(u, v) = \frac{\delta^i_p \delta^j_q}{u - v}, \qquad \beta^{ij}_{pq}(u, v) = \frac{\delta^i_p \delta^j_q}{v - u} + \tilde{r}^{ij}_{pq} \tag{3.46}$$

and

$$r^{ij}_{pq}(u, v) = \frac{\delta^i_q \delta^j_p}{u - v} + \tilde{r}^{ij}_{pq}, \qquad \beta^{ij}_{pq}(u, v) = \frac{\delta^i_q \delta^j_p}{v - u} \tag{3.47}$$

satisfy (3.39). □

Formulas (3.46) and (3.47) generate families of constant trace Poisson brackets [15]. To get them one should set $\mathbf{E}_i(x) = \sum_0^n e_{ij} x^j$ and apply the construction of Proposition 1. As a result, some trace brackets on free associative algebra with mn generators e_{ij} will be defined.

4. Solutions in the case $m = 1$ and their classification.

If $m = 1$ the following solutions of functional equations (3.18), (3.19)

$$r(u, v, x, y) = \frac{1}{u - v} - \frac{1}{x - y}; \tag{4.48}$$

$$r(u, v, x, y) = \frac{e^{u-v} - e^{x-y}}{(e^{u-v} - 1)(e^{x-y} - 1)}; \tag{4.49}$$

$$r(u, v, x, y) = \frac{\theta_{11}(u - v + x - y)}{\theta_{11}(u - v)\theta_{11}(x - y)}; \tag{4.50}$$

where

$$\theta_{11}(u) = \sum_{n\in\mathbb{Z}} (-1)^n \exp\left(\pi i(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})u\right),$$

have been found in [4] and independently in [10]. Here $\theta_{11}(u)$ is the quasi-periodic Jacobi theta-function on the lattice $\mathbb{Z}\oplus\tau\mathbb{Z}$ and $\Im\tau > 0$. Notice that all these solutions have the form $r = r(u-v, x-y)$. For arbitrary m some solutions of the form $r = r(u-v, x, y)$ have been obtained in [12] by pure geometrical methods. Deep general statements concerning the relations of such solutions with quantum $\mathfrak{gl}(m)$ r-matrices, A_∞-constraints and mirror conjecture on elliptic curves have been discovered in [10]. However no relations of these solutions with two-parametric Poisson and double Poisson brackets was not observed.

We present a classification of solutions in the case $m = 1$ up to the following equivalence transformations

$$r(u,v,x,y) \to r(u,v,x,y)\,\frac{q(u,y)q(v,x)}{q(u,x)q(v,y)} \tag{4.51}$$

and

$$u\to\phi(u), \qquad v\to\phi(v), \qquad x\to\psi(x), \qquad y\to\psi(y) \tag{4.52}$$

where q, ϕ, ψ are arbitrary functions.

We claim the following

Theorem 3. *Any solution $r(u,v,x,y)$ of (3.18),(3.19) that has simple poles at $u = v$ and $x = y$ is equivalent to one of solutions (4.48)-(4.50).*

We split the proof in four lemmas:

Lemma 1. Any solution of (3.14), (3.15) can be reduced by a transformation of the form (4.51) to a solution of the following form

$$r(u,v,x,y) = f(u,v) + g(x,y) + h(u,x) - h(v,y), \tag{4.53}$$

where f, g, h are some functions of two variables such that $f(u,v) = -f(v,u)$, $g(x,y) = -g(y,x)$. □

Assume that both f and g have a simple pole at $u = v$ and $x = y$ correspondingly. Substituting (4.53) into (3.15) and expanding the obtained expressions at $y = x$ and at $v = u$, we prove the following statements.

Lemma 2. If f and g have a simple pole at $u = v$ and $x = y$ then up to a transformation (4.52)

$$f(u,v) = \frac{U(u)+U(v)}{u-v}, \qquad g(x,y) = \frac{V(x)+V(y)}{x-y}, \tag{4.54}$$

where

$$U(x) = \sqrt{P}, \quad P(x) = \sum_{i=0}^{4} k_i x^i, \qquad V(x) = \sqrt{Q}, \quad Q(x) = \sum_{i=0}^{4} r_i x^i. \qquad \square \tag{4.55}$$

Consider the case $P \neq 0$, $Q \neq 0$.

Lemma 3. The function h has the form

$$h(u,x) = A_1(u) - A_2(x) + Z(B_1(u) - B_2(x)), \tag{4.56}$$

where

$$A_1' = \frac{-V'^2 - VV''}{6V}, \qquad A_2' = \frac{-U'^2 - UU''}{6U}, \qquad B_1' = \frac{1}{V}, \qquad B_2' = \frac{1}{U},$$

and

$$Z''^2 = 2Z'^3 + \lambda_1 Z' + \lambda_2 \tag{4.57}$$

for some constants λ_i, depending on the coefficients of P and Q. □

Lemma 4. For the polynomials P and Q the functions

$$I_1(P) = k_2^2 - 3k_1k_3 + 12k_0k_4, \qquad I_2(P) = 2k_2^3 - 9k_1k_2k_3 + 27k_0k_3^2 + 27k_1^2k_4 - 72k_0k_2k_4$$

have the same values and

$$\lambda_1 = -\frac{I_1}{6}, \qquad \lambda_2 = \frac{I_2}{108}. \qquad \square \tag{4.58}$$

Lemmas 2-4 describe all solutions of the form (4.53) under assumption $P \neq 0$, $Q \neq 0$. To bring these solutions to a canonical form we may use fraction-linear transformations of the form

$$u \to \frac{a_1u + b_1}{c_1u + d_1}, \quad v \to \frac{a_1v + b_1}{c_1v + d_1}, \qquad x \to \frac{a_2x + b_2}{c_2x + d_2}, \quad y \to \frac{a_2y + b_2}{c_2y + d_2}. \tag{4.59}$$

It is easy to verify that I_1 and I_2 are semi-invariants with respect to (4.59). It is clear that P can be reduced by (4.59) to one of the following canonical forms: $P(x) = 1$, $P(x) = x$, $P(x) = x^2$, $P(x) = x(x-1)$, and $P(x) = x(x-1)(x-\sigma)$.

Moreover, the canonical form $P = x(x-1)$ can be reduced to $P = x^2$ by the transformation $x \to -\frac{(2x-1)^2}{8x}$ while $P = x$ is related to $P = 1$ by $x \to \frac{x^2}{4}$. Thus we may consider only three canonical forms for P: $P = 1$, $P = x^2$ and $P = x(x-1)(x-\sigma)$ and the same canonical forms for Q. Taking into account Lemma 4, we can verify that these canonical forms can be combined in a unique way: 1) $P = Q = 1$; 2) $P = Q = x^2$ and 3) $P = Q = x(x-1)(x-\sigma)$.

The solutions corresponding to the cases 1) and 2) can be written (up to the equivalence) in the following uniform way:

$$r(u,v,x,y) = \frac{p_1uv + p_2(u+v) + p_3}{(u-v)} - \frac{p_1xy + p_2(x+y) + p_3}{(x-y)}.$$

Any such solution is equivalent to (4.48) or to (4.49).

In the case $P(x) = x(x-1)(x-\sigma)$ we perform the transformation $u \to \wp(u), v \to \wp(v), x \to \wp(x), y \to \wp(y)$ and write the solution in the form

$$r = \frac{1}{\theta_{11}'(0} \left(\frac{\theta_{11}'(u-v)}{\theta_{11}(u-v)} - \frac{\theta_{11}'(u+x+\eta)}{\theta_{11}(u+x+\eta)} + \frac{\theta_{11}'(x-y)}{\theta_{11}(x-y)} - \frac{\theta_{11}'(v+y+\eta)}{\theta_{11}(v+y+\eta)} \right),$$

where η is arbitrary parameter. This solution can rewritten as

$$r = \frac{\theta_{11}(u-v+x-y)\theta_{11}(u+y+\eta)\theta_{11}(v+x+\eta)}{\theta_{11}(u-v)\theta_{11}(x-y)\theta_{11}(u+x+\eta)\theta_{11}(v+y+\eta)},$$

which is equivalent to (4.50).

It can be verified that in the case $U \neq 0,\ V = 0$ any solution is equivalent to $r(u,v,x,y) = \frac{1}{u-v}$ while if $U = 0,\ V \neq 0$ we arrive at $r(u,v,x,y) = \frac{1}{x-y}$.

5. Linear parameter-dependent double brackets and Rota-Baxter equations

The general ansatz for local linear double Poisson brackets is given by

$$\begin{aligned} \{\!\{\mathbf{E}_i(u), \mathbf{E}_j(v)\}\!\} = a_{ij}^k(u,v)\mathbf{E}_k(u) \otimes 1 + b_{ij}^k(u,v)\mathbf{E}_k(v) \otimes 1+ \\ c_{ij}^k(u,v) 1 \otimes \mathbf{E}_k(u) + d_{ij}^k(u,v) 1 \otimes \mathbf{E}_k(v) \end{aligned} \tag{5.60}$$

It follows from property (3.22) that $c_{ij}^k(u,v) = -b_{ji}^k(v,u), d_{ij}^k(u,v) = -a_{ji}^k(v,u)$. The (twisted) Jacobi identity (3.23) is equivalent to the following functional relations for the coefficients:

$$\begin{aligned} a_{ij}^\sigma(u,v)a_{k\sigma}^\lambda(w,u) - a_{ki}^\sigma(w,u)a_{\sigma j}^\lambda(w,v) + a_{k\sigma}^\lambda(w,v)b_{ij}^\sigma(u,v) = 0; \\ b_{ij}^\sigma(u,v)b_{k\sigma}^\lambda(w,v) - b_{ki}^\sigma(w,u)b_{\sigma j}^\lambda(u,v) - b_{\sigma j}^\lambda(w,v)a_{ki}^\sigma(w,u) = 0; \\ a_{jk}^\sigma(v,w)b_{i\sigma}^\lambda(u,v) - b_{ij}^\sigma(u,v)a_{\sigma k}^\lambda(v,w) = 0. \end{aligned} \tag{5.61}$$

Remark 6. The same conditions are equivalent to the associativity of the following product:

$$\mathbf{E}_i(u)\mathbf{E}_j(v) = a_{ij}^k(u,v)\mathbf{E}_k(u) + b_{ij}^k(u,v)\mathbf{E}_k(v).$$

In the case $m = 1$ we get the following two functional equations to a and b:

$$a(u,v)a(w,u) - a(w,u)a(w,v) + a(w,v)b(u,v) = 0;$$

$$b(u,v)b(w,v) - b(w,u)b(u,v) - b(w,v)a(w,u) = 0.$$

If $ab \neq 0$, the general answer for these equations is of the form

$$a(u,v) = \frac{a_1(v)b_1(u)}{b_1(u) - b_1(v)}, \qquad b(u,v) = \frac{a_1(u)b_1(v)}{b_1(v) - b_1(u)},$$

where a_1, b_1 are arbitrary functions of one variable.

The simplest solution of (5.61) for arbitrary m is given by

$$a_{ij}^k(u,v) = \frac{c_{ij}^k}{u-v}, \qquad b_{ij}^k(u,v) = \frac{c_{ij}^k}{v-u}, \tag{5.62}$$

where c_{ij}^k are structure constants of any m-dimensional associative algebra $\mathcal{A}$.

More general, let $r(u,v) : \mathcal{A} \to \mathcal{A}$ be any solution of the parameter dependent Rota-Baxter equation

$$(r(u,w)x)(r(u,v)y) - r(u,v)((r(v,w)x)y) - r(u,w)(x(r(w,v)y)) = 0 \qquad (5.63)$$

introduced in [3]. Then

$$a^k_{ij}(u,v) = c^k_{is} r^s_j(u,v), \qquad b^k_{ij}(u,v) = c^k_{sj} r^s_i(v,u)$$

satisfy (5.61). Some solutions of equation (5.63) have been found in [3].

In the case $\mathcal{A} = \mathrm{Mat}_m(\mathbb{C})$ the Rota-Baxter equation (5.63) has the form

$$r^{\alpha p}_{\beta j}(u,v) r^{r\sigma}_{sp}(u,w) = r^{rp}_{s\beta}(v,w) r^{\alpha\sigma}_{pj}(u,v) + r^{p\sigma}_{sj}(u,w) r^{\alpha r}_{\beta p}(w,v). \qquad (5.64)$$

This equation is a partial case of (2.7) under the additional skew-symmetry assumption (2.6):

$$r^{jm}_{ik}(u,v) = -r^{mj}_{ki}(v,u).$$

.

Remark 7. The parameter dependent Rota-Baxter equation (5.63) is an associativity condition of the second multiplication operation

$$x \circ_{r(u,v)} y := r(u,v)(x)y + xr(u,v)(y)$$

in the algebra $\mathcal{A}$ which becomes *doubly associative algebra* in terminology of M. Semenov-Tyan-Shansky (who had proposed the first example of generalized parameter-dependent Rota-Baxter equation in [2]. His example deals with the associative algebra $\mathcal{A} = \mathrm{Mat}_n \otimes \mathbb{C}[u, u^{-1}]$ of matrices over the Laurent polynomial ring. His operator $r(u,v)$ is defined as identity on the sub-algebra of matrix with non-negative degrees in u and minus identity on the sub-algebra of matrix with strictly negative degrees in u. This operator satisfies the special case of general Rota-Baxter weight λ relation, (here λ is a complex parameter):

$$(r(u,w)x)(r(u,v)y) - r(u,v)((r(v,w)x)y) - r(u,w)(x(r(w,v)y)) = \lambda xy. \qquad (5.65)$$

The equation (5.63) corresponds the case $\lambda = 0$ and the Semenov-Tian-Shansky Rota-Baxter operator corresponds to $\lambda = -1$. [c]

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