Integrable systems associated with elliptic algebras

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INTEGRABLE SYSTEMS ASSOCIATED WITH ELLIPTIC ALGEBRAS

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Abstract. We construct some new Integrable Systems (IS) both classical and quantum associated with elliptic algebras. Our constructions are partly based on the algebraic integrability mechanism given by the existence of commuting families in skew fields and partly - on the internal properties of the elliptic algebras and their representations. We give some examples to make an evidence how these IS are related to previously studied.

Introduction. This paper is an attempt to establish a direct connection between two close subjects of modern Mathematical Physics - the theory of Integrable Systems (IS) and the Elliptic Algebras. The aim of this connection is two-fold: we clarify some our recent results in the both domains and fill the natural gap proving that some large class of the Elliptic Algebras carries in fact the structure of an IS.

We will start with a short account in the subject of the story and will describe briefly a type of the IS's under cosideration.

In [12], B.Enriquez and second author had proposed a construction of commuting families of elements in skew fields. They explained how to use this construction in Poisson fraction fields to give an another proof of the integrability of Beauville - Mukai integrable systems associated with a K3 surface S ([1]).

The Beauville-Mukai systems had appeared as the Lagrangian fibrations which have the form $S^{[g]} \to |\mathcal{L}| = \mathbb{P}(H^0(S,\mathcal{L}))$, where $S^{[g]}$ is the Hilbert scheme of g points of S, equipped with a symplectic structure introduced in [19], and $\mathcal{L}$ is a line bundle on S. Later, the authors of [8] explained that these systems are natural deformations of the "separated" (in the sense of [13]) versions of Hitchin's integrable systems, more precisely, of their description in terms of spectral curves (already present in [15]). Beauville-Mukai systems can be generalized to surfaces with a Poisson structure (see [3]). When S is the canonical cone Cone(C) of an algebraic curve C these systems coincide with the separated version of Hitchin's systems. The paper [12] shows how the commuting families construction provides a quantization of these systems on the canonical cone.

A quantization of Hitchin's system was proposed in [2]. It seems interesting to construct quantization of Beauville-Mukai systems and to compare it with Beilinson - Drinfeld quantization. We had conjectured the correspondence between the results of [2] and a quantization of fraction fields in [9]. A part of this program was realized in the [9] for the case of $S = T^*\mathbb{P}^1_n$, where $\mathbb{P}^1_n$ means a rational curve with n marked points.

Another main object of the paper is the families of Elliptic Algebras. These algebras (with 4 generators) were appeared in the works of Sklyanin [20],[21] and then they were generalized (for any number of generators) and intensively studied by Feigin and one of the authors([22],[27],[28]). These algebras can be considered as a flat deformation of the polynomial rings such that the linear (in the deformation parameter) term is given by the quadratic Poisson brackets. The geometric meaning of the Poisson structures was underscored in [29](see also [41] and [36]) - it is the Poisson structures of moduli spaces of holomorphic bundles on the elliptic curve. We will use the recent survey [30] as a main source of the results and references in the theory of elliptic algebras.

The relevance of Elliptic Algebras to the theory of IS was manifested from the very beginning. They appeared in Sklyanin's demonstration of the integrability of the Landau-Lifshitz model ([20],[21])in the frame of the Faddeev's school ideology (Quantum Inverse Scattering Method and $R-$ matrix approach). Later Cherednik had observed the relation of the Elliptic Algebras defined in [22] with the Belavin $R-$matrix (see [32]). An interesting observation of Krichever and Zabrodin giving an interpretation of a generator in the Sklyanin's Elliptic Algebra as a Hamiltonian of 2-point Ruijsenaars IS ([16]) was generalized later in [4] to the case of double-elliptic 2-point classical model. However, all applications of these algebras to the IS theory had an indirect character until the last two years.

We should mention also the results which were obtained in the paper of Sokolov-Tsyganov ([34]) where they construct,(using the Sklyanin's definition of quadratic Poisson structures), some classical commuting families associated with this Poisson algebras. The integrability of these families is implied by the Sklyanin's ideology of Separated Variables (SoV) and technically is based on some generalization of the classical methods going back to Jacobi, Liouville and Stäckel (which ideologically is very close to the classical part of the theorems in [12] and [35]). However, all their results are in the "non-elliptic" case.

Recently, an example of integrable systems associated with linear and quadratic Poisson brackets given by the elliptic Belavin-Drinfeld classical r-matrix was proposed in [18]. This system (an elliptic rotator) appears both in finite and infinite-dimensional cases. They give an elliptic version of $2D$ ideal hydrodynamics on the symplectomorphism group of the 2-dimensional torus as well as on a non-commutative torus. We should also mention another appearance of the elliptic algebras in the context of Non-Commutative geometry (see [40]). It would be interesting to relate them to a numerous modern attempts to define a Non-Commutative version of IS theory.

In this paper we construct some IS which appear directly in the frame of the elliptic algebras. The elliptic algebras are figured here in two-fold way: sometimes, they are carrying the commuting families of Hamiltonians, sometimes, - they provide a necessary background to our constructions which use their properties and representation theory (basically, the so-called "functional realization" and the "bosonization" mappings).

A family of coordinated (compatible) elliptic Poisson structures was introduced in [31]. This family contains three quadratic Poisson brackets such that their generic linear combination is the quasi-classical limit $q_n(\mathcal{E})$ of an elliptic algebra $Q_n(\mathcal{E}, \eta)$. The famous Lenard - Magri scheme provides the existence of a classical integrable system associated with the elliptic curve but it was not clear how to get a quantum counterpart of this system because of lack of the knowledge how to quantize in general the Magri-Lenard scheme. Nevertheless, this system can be quantized for $n = 2m$ using the approach developing in [12] and it is one of the main results of our paper.

Some of the "elliptic" commuting elements which we construct in this article are related to the quantum version of the above-mentioned bi-hamiltonian system. Some other families are associated with a special choice of the elliptic algebra. These families are obtained, in one hand, as the direct application of the construction from [12] to the elliptic algebras and, in other hand, by using the properties of the "bosonization" homomorphism, constructed in earlier works ([27],[28]). Some of these families (under the appropriate choice of their numeric parameters) may be interpreted as examples of algebraic completely integrable systems. We give a geometric interpretation to some of them describing a link with the Lagrangian fibrations on symmetric products of elliptic curve cone, giving a version of the Beauville-Mukai systems (see [1],[13],[12],[39]).

Roughly speaking, the integrable systems of the first type have as the phase space a $2m-$ dimensional component of the moduli space of parabolic rank two bundles on the given elliptic curve $\mathcal{E}$. More precisely, the coordinate ring of the open dense part of the latter has a structure of a quadratic Poisson algebra isomorphic to $q_{2m}(\mathcal{E})$. We have explicitly verified that the quantum commuting elements from our construction are the same as the latter obtained from the Lenard-Magri scheme for $m = 3$ (the first non-trivial case).

Our main theorem (Thm.3.1) takes place for $n = 2m$, but we are sure that there are some interesting integrable quantum systems in the case of $n = 2m+1$. It would be interesting to study the bi-hamiltonian structures giving the algebra $q_{2m+1}(\mathcal{E})$ using the results of Gelfand-Zakharevich ([33]) on the geometry of bi-hamiltonian systems in the case of impair-dimensional Poisson manifolds. The precise quantum version of these systems in the context of the elliptic algebras $Q_{2m+1}(\mathcal{E}, \eta)$ is still obscure and should be a subject of further studies.

The theorems from [12] may be also interpreted as an algebraic version of the SoV method (as it was argued in [35]). Hence, it is very plausible that some of our quantum commuting families arising from the generalization of the Jacobi-Liouville-Stäckel conditions (which are guaranteed by existence conditions of the

Cartier - Foata NC determinants) are the quantum elliptic versions of the IS from [34]. We hope to return to this question in our future paper.

We give also some low-dimensional examples of our construction.

1. Commuting families in some non-commutative algebras

Let A be an associative algebra with unit. We will suppose further that we work with a skew field.

Fix a natural number $n \geq 2$. We will suppose that there are n subalgebras $B_n \subset A$ in A such that for any couple of the indices $i \neq j, 1 \leq i, j \leq n$ the elements $b_{(i)} \in B_i$ and $b_{(j)} \in B_j$ are commuting (and inside of the each subalgebra B_i the elements generically are not commuting).

Let us consider the following data: take an $n \times (n+1)$ matrix $\mathcal{M}$

$$\begin{pmatrix} b^0_{(1)} & b^1_{(1)} & \dots & b^{n-1}_{(1)} & b^n_{(1)} \\ b^0_{(2)} & b^1_{(2)} & \dots & b^{n-1}_{(2)} & b^n_{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b^0_{(n)} & b^1_{(n)} & \dots & b^{n-1}_{(n)} & b^n_{(n)} \end{pmatrix} \tag{1}$$

such that all elements of $i-$th row belong to the $i-$th subalgebra $B_{(i)}$.

We will denote by $\mathcal{M}^i$ a $n \times n$ matrix which is resulted from the matrix $\mathcal{M}$ with the $i-th$ row erased. The corresponding determinants (if they exist) will be denote by M^i.

Now we observe that all principal $n+1$ minors

$$M^0, M^1, \dots, M^n$$

of the highest order n are correctly defined with the help of *Cartier - Foata non-commutative determinant.*

It's definition repeats *verbatim* the standard one because in all n matrices

$$\mathcal{M}^0, \mathcal{M}^1, \dots, \mathcal{M}^n$$

the entries in the *different rows* are commute and each summand in the standard definition of the determinant contains the product of n elements of different rows whose product is order - independent.

The following theorem was proved in [12]

Theorem 1.1. *Suppose that one of the matrix $\mathcal{M}^i, i = 1, \dots, n$, say, $\mathcal{M}^0$, is invertible. Then the elements $H_i = (M^0)^{-1}M^i, i = 1, \dots, n$ are commute in the skew field A (which we are identifying with its image in the fraction field* Frac(A) *under the monomorphic embedding $A \to$* Frac(A).

The proof of the theorem is achieved by some tedious but a straightforward induction procedure.

The similar results were obtained in the framework of so-called multi-parametric spectral problems in Operator Analysis ([38]) and in the framework of so called

Seiberg-Witten integrable systems associated with a hyperelliptic spectral curves in [35].

The important step in the demonstration of the 1.1 is the following *"triangle"* relations which are similar to the usual Yang -Baxter relation:

$$M^i(M^0)^{-1}M^j = M^j(M^0)^{-1}M^i \tag{2}$$

$$B^{ij}(M^0)^{-1}B^{kj} = B^{ik}(M^0)^{-1}B^{ij}, \tag{3}$$

where B^{ij} is the co-factor of the matrix element $b^i_j,\ 0 \le i,j \le n$.

The theorem 1.1 can be re-formulated to give the following result

Corollary 1.1. *Let A be an algebra, $(f_{i,j})_{0\le i\le n, 1\le j\le n}$ be elements of A such that*

$$f_{i,j}f_{k,\ell} = f_{k,\ell}f_{i,j}$$

for any i,j,k,ℓ such that $j \neq \ell$. For any $I \subset \{0,\dots,n\}, J \subset \{1,\dots,n\}$ of the same cardinality, we set $\Delta_{I,J} = \sum_{\sigma\in \mathrm{Bij}(I,J)} \epsilon(\sigma) f_{i,\sigma(i)}$. Here $\mathrm{Bij}(I,J)$ denotes a set of bijections between two sets of the indices I and J. Assume that the $\Delta_{I,J}$ are all invertible. Set $\Delta_i = \Delta_{\{1,\dots,n\},\{0,\dots,\check{i},\dots,n\}}$ to be the element corresponded to J with the index i omitted. Then the

$$H_i = (\Delta_0)^{-1}\Delta_i$$

all commute together.

1.1. **Poisson commuting families.** We will fix a base field $\mathbf{k}$ of characteristic $\neq 2$. The following observation is straightforward:

Lemma 1.1. *If B is an integral Poisson algebra, then there is a unique Poisson structure on* $\mathrm{Frac}(B)$ *extending the Poisson structure of B.*

This structure is uniquely defined by the relations

$$\{1/f, g\} = -\{f,g\}/f^2, \{1/f, 1/g\} = \{f,g\}/(f^2g^2)$$

.

Theorem 1.1 has a Poisson counterpart.

Theorem 1.2. *Let A be a Poisson algebra. Assume that A is integral, and let $\pi : A \hookrightarrow \mathrm{Frac}(A)$ be its injection in its fraction field. For each $n-$uple of Poisson subalgebras $B_1,\dots,B_n$ of A such that the elements of pair-wise different subalgebras B_i are Poisson commuting (for any pair of indices $i,j,i\neq j$ the elements $b_i\in B_i$ and $b_j \in B_j$ we have $\{b_i,b_j\} = 0$.) We will write the analogue of the matrix (1) like the vector-row: $\mathcal{M} = [b^0, b^1,\dots,b^n]$, where*

$$b^i = \begin{pmatrix} b^i_1 \\ b^i_2 \\ \vdots \\ b^i_n \end{pmatrix}.$$

We set

$$\Delta_i^{\text{class}} = \det[b^0, \dots, \check{b}^i, \dots, b^n].$$

Here as usual we denote by $\check{b}^i$ *the* $i-$*th omitted column. Then if* Δ_0^{class} *is nonzero we set* $H_i^{\text{class}} = \Delta_i^{\text{class}}/\Delta_0^{\text{class}}$ *and the family* $(H_i^{\text{class}})_{i=1,\dots,n}$ *is Poisson-commutative:*

$$\{H_i^{\text{class}}, H_j^{\text{class}}\} = 0$$

for any pair (i,j).

Remark 1. The elements b_i^k and b_j^l of the matrix $\mathcal{M}$ belongs to the different subalgebras B_i and B_j if $i \neq j$ and hence are Poisson-commute. This condition reminds the classical constrains on the Poisson brackets between matrix elements appeared in XIX century in the papers of Stäckel on the Separation of Variables of Hamilton-Jacobi systems ([37]). So our theorem can be considered as an algebraic re-definition of the Stäckel conditions

1.1.1. *Plucker relations.* We want to remind the important step of the second proof in [12] which shows the relations between the commuting elements and the Plucker-like equations.

We have to prove

$$\Delta_i^{\text{class}}\{\Delta_j^{\text{class}}, \Delta_k^{\text{class}}\} + \text{cyclic permutation in } (i,j,k) = 0. \tag{4}$$

We have

$$\Delta_i^{\text{class}} = \sum_{p=1}^{n}\sum_{\alpha=0}^{n}(-1)^{p+\alpha}(b^\alpha)^{(p)}(\Delta_{\alpha,i})^{(1\dots\check{p}\dots n)},$$

where (if $\alpha \neq i$)

$$\Delta_{\alpha,i}^{(1\dots\check{p}\dots n)} = (-1)^{1_{\alpha<i}}\det[b^0\dots\check{b}^\alpha\dots\check{b}^i\dots b^n]^{(p)}$$

(which means that the $p-$th row in the matrix $[b^0,\dots,\check{b}^\alpha,\dots,\check{b}^i\dots b^n]$ should be erased.)

We set $1_{\alpha<i} = 1$ if $\alpha < i$ and 0 otherwise. If $\alpha = i$ we assume $\Delta_{\alpha,i} = 0$. Now we have

$$\{\Delta_i^{\text{class}}, \Delta_j^{\text{class}}\} = \sum_{p=1}^{n}\sum_{\alpha,\beta=0}^{n}(-1)^{\alpha+\beta}(\{b^\alpha, b^\beta\})^{(p)}\left(\Delta_{\alpha,i}\Delta_{\beta,j} - \Delta_{\beta,i}\Delta_{\alpha,j}\right)^{(1\dots\check{p}\dots n)},$$

so identity (4) is a consequence of

$$\forall(i,j,k,\alpha,\beta,\gamma), \quad \sum_{\sigma\in\text{Perm}(i,j,k)}\epsilon(\sigma)\Delta_{\alpha,\sigma(i)}\Delta_{\beta,\sigma(j)}\Delta_{\gamma,\sigma(k)} = 0. \tag{5}$$

When $\text{card}\{\alpha,\dots,k\} = 3$, this identity follows from the antisymmetry relation $\Delta_{i,j} + \Delta_{j,i} = 0$.

When $\text{card}\{\alpha,\dots,k\} = 4$ (resp., $5, 6$), it follows from the following Plucker identities (to get (5), one should set $V = (A^{\otimes n})^{\oplus n}$ and Λ some partial determinant).

Let V be a vector space. Then
– if $\Lambda \in \wedge^2(V)$, and $a, b, c, d \in V$, then

$$\Lambda(a,b)\Lambda(c,d) - \Lambda(a,c)\Lambda(b,d) + \Lambda(a,d)\Lambda(b,c) = 0; \tag{6}$$

– if $\Lambda \in \wedge^3(V)$ and $a, b, c, b', c' \in V$, then

$$\begin{aligned}&\Lambda(b,c,c')\Lambda(a,c,b')\Lambda(b,b',c') + \Lambda(b,c,b')\Lambda(c,b',c')\Lambda(a,b,c')\\ &- \Lambda(b,c,b')\Lambda(a,c,c')\Lambda(b,b',c') - \Lambda(b,c,c')\Lambda(c,b',c')\Lambda(a,b,b') = 0;\end{aligned}$$

– if $\Lambda \in \wedge^4(V)$ and $a, b, c, a', b', c' \in V$, then

$$\begin{aligned}&\Lambda(b,c,b',c')\Lambda(a,c,a',c')\Lambda(a,b,a',b') + \Lambda(b,c,a',c')\Lambda(a,c,a',b')\Lambda(a,b,b',c')\\ &+ \Lambda(b,c,a',b')\Lambda(a,c,b',c')\Lambda(a,b,a',c')\\ &- \Lambda(b,c,b',c')\Lambda(a,c,a',b')\Lambda(a,b,a',c') - \Lambda(b,c,a',b')\Lambda(a,c,a',c')\Lambda(a,b,b',c')\\ &- \Lambda(b,c,a',c')\Lambda(a,c,b',c')\Lambda(a,b,a',b') = 0.\end{aligned} \tag{7}$$

We refer to [12] for the proof of these identities. We will need them below in some special situation arising with the commuting elements in associative and Poisson algebras which are directly connected with elliptic curves and vector bundles on them. This Plucker relations can be interpreted as a kind of Riemann-Fay identities which are in its turn related to integrable (difference) equations in Hirota bilinear form.

2. Elliptic algebras

Now we should describe one of the main heroes of our story - the elliptic algebras. We will follow to the survey [30] in our notations and also we will refer to it as a main source of the results and its proofs in this section.

2.1. **Definition and the main properties.** The elliptic algebras are the associative quadratic algebras $Q_{n,k}(\mathcal{E}, \eta)$ which were introduced in the papers [22, 28]. Here $\mathcal{E}$ is an elliptic curve and n, k are integer numbers without common divisors ,such that $1 \le k < n$ while η is a complex number and $Q_{n,k}(\mathcal{E}, 0) = \mathbb{C}[x_1, ..., x_n]$.

Let $\mathcal{E} = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}, \tau \in \mathbb{C}, \Im\tau > 0$. The algebra $Q_{n,k}(\mathcal{E}, \eta)$ has generators $x_i, i \in \mathbb{Z}/n\mathbb{Z}$ subjected to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0$$

and have the following properties: 1) $Q_{n,k}(\mathcal{E}, \eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus ...$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$, here $*$ denotes the algebra multiplication. In other words, the algebras $Q_{n,k}(\mathcal{E}, \eta)$ are $\mathbb{Z}$ - graded;

2) The Hilbert function of $Q_{n,k}(\mathcal{E},\eta)$ is $\sum_{\alpha\geq 0}\dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.

We consider here the theta-functions $\theta_i(z), i=1,\dots,n$ as a base in the space of the theta-functions $\Theta_n(\Gamma)$ of the order n which are subordinated to the following relations of quasi-periodicity

$$\theta_i(z+1)=\theta_i(z),\ \theta_i(z+\tau)=(-1)^n\exp(-2\pi\sqrt(-1)nz)\theta_i(z), i=0,\dots,n-1.$$

The theta-function of the order 1 $\theta(z)\in\Theta_1(\Gamma)$ satisfies to the conditions $\theta(0)=0$ and $\theta(-z)=\theta(z+\tau)=-\exp(-2\pi\sqrt(-1)z)\theta(z)$.

We see that the algebra $Q_{n,k}(\mathcal{E},\eta)$ for fixed $\mathcal{E}$ is a flat deformation of the polynomial ring $\mathbb{C}[x_1,...,x_n]$. The linear (in η)term of this deformation gives rise to a quadratic Poisson algebra $q_{n,k}(\mathcal{E})$.

The geometric meaning of the algebras $Q_{n,k}$ was underscored in [29],[41] where it was shown that the quadratic Poisson structure in the algebras $q_{n,k}(\mathcal{E})$ associated with the above-mentioned deformation is nothing but the Poisson structure on $\mathbb{P}^{n-1}=\mathbb{P}Ext^1(E,\mathcal{O})$, where E is a stable vector bundle of rank k and degree n on the elliptic curve $\mathcal{E}$.

In what follows we will denote the algebras $Q_{n,1}(\mathcal{E},\eta)$ by $Q_n(\mathcal{E},\eta)$

2.2. **Algebra $Q_n(\mathcal{E},\eta)$.**

2.2.1. *Construction.* For any $n\in\mathbb{N}$, any elliptic curve $\mathcal{E}=\mathbb{C}/\Gamma$, and any point $\eta\in\mathcal{E}$ we construct a graded associative algebra $Q_n(\mathcal{E},\eta)=\mathbb{C}\oplus F_1\oplus F_2\oplus\dots$, where $F_1=\Theta_n(\Gamma)$ and $F_\alpha=S^\alpha\Theta_n(\Gamma)$. By construction, $\dim F_\alpha=\frac{n(n+1)\dots(n+\alpha-1)}{\alpha!}$. It is clear that the space F_α can be realized as the space of holomorphic symmetric functions of α variables $\{f(z_1,\dots,z_\alpha)\}$ such that

$$\begin{aligned}&f(z_1+1,z_2,\dots,z_\alpha)=f(z_1,\dots,z_\alpha),\\&f(z_1+\tau,z_2,\dots,z_\alpha)=(-1)^ne^{-2\pi inz_1}f(z_1,\dots,z_\alpha).\end{aligned}\tag{8}$$

For $f\in F_\alpha$ and $g\in F_\beta$ we define the symmetric function $f*g$ of $\alpha+\beta$ variables by the formula

$$f*g(z_1,\dots,z_{\alpha+\beta})=\frac{1}{\alpha!\beta!}\sum_{\sigma\in S_{\alpha+\beta}}f(z_{\sigma_1}+\beta\eta,\dots,z_{\sigma_\alpha}+\beta\eta)g(z_{\sigma_{\alpha+1}}-\alpha\eta,\dots,z_{\sigma_{\alpha+\beta}}-\alpha\eta)\times$$

$$\times\prod_{\substack{1\leq i\leq\alpha\\ \alpha+1\leq j\leq\alpha+\beta}}\frac{\theta(z_{\sigma_i}-z_{\sigma_j}-n\eta)}{\theta(z_{\sigma_i}-z_{\sigma_j})}.\tag{9}$$

In particular, for $f,g\in F_1$ we have

$$f*g(z_1,z_2)=f(z_1+\eta)g(z_2-\eta)\frac{\theta(z_1-z_2-n\eta)}{\theta(z_1-z_2)}+f(z_2+\eta)g(z_1-\eta)\frac{\theta(z_2-z_1-n\eta)}{\theta(z_2-z_1)}.$$

Here $\theta(z)$ is a theta function of order one.

Proposition 2.1. *If $f \in F_\alpha$ and $g \in F_\beta$, then $f * g \in F_{\alpha+\beta}$. The operation $*$ defines an associative multiplication on the space $\oplus_{\alpha\geq 0} F_\alpha$*

2.2.2. *Main properties of the algebra $Q_n(\mathcal{E},\eta)$.* By construction, the dimensions of the graded components of the algebra $Q_n(\mathcal{E},\eta)$ coincide with those for the polynomial ring in n variables. For $\eta = 0$ the formula for $f * g$ becomes

$$f * g(z_1,\dots,z_{\alpha+1}) = \frac{1}{\alpha!\beta!}\sum_{\sigma\in S_{\alpha+\beta}} f(z_{\sigma_1},\dots,z_{\sigma_\alpha})g(z_{\sigma_{\alpha+1}},\dots,z_{\sigma_{\alpha+\beta}}).$$

This is the formula for the ordinary product in the algebra $S^*\Theta_n(\Gamma)$, that is, in the polynomial ring in n variables. Therefore, for a fixed elliptic curve $\mathcal{E}$ (that is, for a fixed modular parameter τ) the family of algebras $Q_n(\mathcal{E},\eta)$ is a deformation of the polynomial ring. In particular, there is a Poisson algebra, which we denote by $q_n(\mathcal{E})$. One can readily obtain the formula for the Poisson bracket on the polynomial ring from the formula for $f * g$ by expanding the difference $f * g - g * f$ in the Taylor series with respect to η. It follows from the semi-continuity arguments that the algebra $Q_n(\mathcal{E},\eta)$ with generic η is determined by n generators and $\frac{n(n-1)}{2}$ quadratic relations. One can prove (see §2.6 in [30]) that this is the case if η is not a point of finite order on $\mathcal{E}$, that is, $N\eta \notin \Gamma$ for any $N \in \mathbb{N}$.

The space $\Theta_n(\Gamma)$ of the generators of the algebra $Q_n(\mathcal{E},\eta)$ is endowed with an action of a finite group $\widetilde{\Gamma_n}$ which is a central extension of the group $\Gamma/n\Gamma$ of points of order n on the curve $\mathcal{E}$. It immediately follows from the formula for the product $*$ that the corresponding transformations of the space $F_\alpha = S^\alpha\Theta_n(\Gamma)$ are automorphisms of the algebra $Q_n(\mathcal{E},\eta)$.

2.2.3. *Bosonization of the algebra $Q_n(\mathcal{E},\eta)$.* The main approach to obtain representations of the algebra $Q_n(\mathcal{E},\eta)$ is to construct homomorphisms from this algebra to other algebras with simple structure (close to Weil algebras) which have a natural set of representations. These homomorphisms are referred to as bosonizations, by analogy with the known constructions of quantum field theory.

Let $B_{p,n}(\eta)$ be a $\mathbb{Z}^p$-graded algebra whose space of degree $(\alpha_1,\dots,\alpha_p)$ is of the form $\{f(u_1,\dots,u_p)e_1^{\alpha_1}\dots e_p^{\alpha_p}\}$, where f ranges over the meromorphic functions of p variables and $e_1,\dots,e_p$ are elements of the algebra $B_{p,n}(\eta)$. Let $B_{p,n}(\eta)$ be generated by the space of meromorphic functions $f(u_1,\dots,u_p)$ and by the elements $e_1,\dots,e_p$ with the defining relations

$$\begin{gathered} e_\alpha f(u_1,\dots,u_p) = f(u_1-2\eta,\dots,u_\alpha+(n-2)\eta,\dots,u_p-2\eta)e_\alpha, \\ e_\alpha e_\beta = e_\beta e_\alpha, \quad f(u_1,\dots,u_p)g(u_1,\dots,u_p) = g(u_1,\dots,u_p)f(u_1,\dots,u_p) \end{gathered} \tag{10}$$

We note that the subalgebra of $B_{p,n}(\eta)$ consisting of the elements of degree $(0,\dots,0)$ is the commutative algebra of all meromorphic functions of p variables with the ordinary multiplication.

Proposition 2.2. *Let $\eta \in \mathcal{E}$ be a point of infinite order. For any $p \in \mathbb{N}$ there is a homomorphism $\phi_p \colon Q_n(\mathcal{E},\eta) \to B_{p,n}(\eta)$ that acts on the generators of the algebra $Q_n(\mathcal{E},\eta)$ by the formula :*

$$\phi_p(f) = \sum_{1\le\alpha\le p} \frac{f(u_\alpha)}{\theta(u_\alpha - u_1)\dots\theta(u_\alpha - u_p)} e_\alpha. \tag{11}$$

Here $f \in \Theta_n(\Gamma)$ is a generator of $Q_n(\mathcal{E},\eta)$ and the product in the denominator is of the form $\prod_{i\neq\alpha}\theta(u_\alpha - u_i)$.

2.2.4. *Symplectic leaves.* We recall that $Q_n(\mathcal{E},0)$ is the polynomial ring $S^*\Theta_n(\Gamma)$. For a fixed elliptic curve $\mathcal{E} = \mathbb{C}/\Gamma$ we obtain the family of algebras $Q_n(\mathcal{E},\eta)$, which is a flat deformation of the polynomial ring. We denote the corresponding Poisson algebra by $q_n(\mathcal{E})$. We obtain a family of Poisson algebras, depending on $\mathcal{E}$, that is, on the modular parameter τ. Let us study the symplectic leaves of this algebra. To this end, we note that, when passing to the limit as $\eta \to 0$, the homomorphism ϕ_p of associative algebras gives a homomorphism of Poisson algebras. Namely, let us denote by $b_{p,n}$ the Poisson algebra formed by the elements $\sum_{\alpha_1,\dots,\alpha_p\ge 0} f_{\alpha_1,\dots,\alpha_p}(u_1,\dots,u_p)e_1^{\alpha_1}\dots e_p^{\alpha_p}$, where $f_{\alpha_1,\dots,\alpha_p}$ are meromorphic functions and the Poisson bracket is

$$\{u_\alpha,u_\beta\} = \{e_\alpha,e_\beta\} = 0; \quad \{e_\alpha,u_\beta\} = -2e_\alpha; \quad \{e_\alpha,u_\alpha\} = (n-2)e_\alpha,$$

where $\alpha \neq \beta$.

The following assertion results from Proposition 6 in the limit as $\eta \to 0$.

Proposition 2.3. *There is a Poisson algebra homomorphism $\psi_p \colon q_n(\mathcal{E}) \to b_{p,n}$ given by the following formula: if $f \in \Theta_n(\Gamma)$, then*

$$\psi_p(f) = \sum_{1\le\alpha\le p} \frac{f(u_\alpha)}{\theta(u_\alpha - u_1)\dots\theta(u_\alpha - u_p)} e_\alpha$$

.

Let $\{\theta_i(u); i \in \mathbb{Z}/n\mathbb{Z}\}$ be a basis of the space $\Theta_n(\Gamma)$ and let $\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}$ be the corresponding basis in the space of elements of degree one in the algebra $Q_n(\mathcal{E},\eta)$ (this space is isomorphic to $\Theta_n(\Gamma)$). For an elliptic curve $\mathcal{E} \subset \mathbb{P}^{n-1}$ embedded by means of theta functions of order n (this is the set of points with the coordinates $(\theta_0(z) : \dots : \theta_{n-1}(z))$) we denote by $C_p\mathcal{E}$ the variety of p-chords, that is, the union of projective spaces of dimension $p-1$ passing through p points of $\mathcal{E}$. Let $K(C_p\mathcal{E})$ be the corresponding homogeneous manifold in $\mathbb{C}^n$. It is clear that $K(C_p\mathcal{E})$ consists of the points with the coordinates

$$x_i = \sum_{1\le\alpha\le p} \frac{\theta_i(u_\alpha)}{\theta(u_\alpha - u_1)\dots\theta(u_\alpha - u_p)} e_\alpha$$

, where $u_\alpha, e_\alpha \in \mathbb{C}$.

Let $2p < n$. Then one can show that $\dim K(C_p\mathcal{E}) = 2p$ and $K(C_{p-1}\mathcal{E})$ is the manifold of singularities of $K(C_p\mathcal{E})$. It follows from Proposition 7 and from the fact that the Poisson bracket is non-degenerate on $b_{p,n}$ for $2p < n$ and $e_\alpha \neq 0$ that the non-singular part of the manifold $K(C_p\mathcal{E})$ is a $2p$- dimensional symplectic leaf of the Poisson algebra $q_n(\mathcal{E})$.

Let n be odd. One can show that the equation defining the manifold $K(C_{\frac{n-1}{2}}\mathcal{E})$ is of the form $C = 0$, where C is a homogeneous polynomial of degree n in the variables x_i. This polynomial is a central function of the algebra $q_n(\mathcal{E})$.

Let n be even. The manifold $K(C_{\frac{n-2}{2}}\mathcal{E})$ is defined by equations $C_1 = 0$ and $C_2 = 0$, where $\deg C_1 = \deg C_2 = n/2$. The polynomials C_1 and C_2 are central in the algebra $q_n(\mathcal{E})$.

3. Integrable systems

There are (at least two) different ways how to construct some commuting elements and IS associated with the elliptic algebras. We will start with the general statements about the commuting elements arising from the ideas and constructions of the section 1 .

3.1. **Commuting elements in the algebras $Q_n(\mathcal{E};\eta)$.** Let us consider the following Weyl-like algebra $\mathcal{V}_n$ with the set of generators $f_1, \dots, f_n, z_1, \dots, z_n$ subjected to the relations

$$0 = [f_i, f_j] = [z_i, z_j] = [f_i, z_j] \ (i \neq j), \ f_i z_i = (z_i - n\eta) f_i.$$

We assume the following commutation relations between the functions in the variables z_i and the elements f_j:

$$f_j F(z_1, \dots, z_n) = F(z_1, \dots, z_j - n\eta, \dots, z_n) f_j.$$

We should precise that the algebra $\mathcal{V}_n$ is spanned as a vector space by the elements of the form $F(z_1, \dots, z_n) f_1^{m_1} \dots f_n^{m_n}$, where F is a meromorphic function in n variables.

Remark 2. We should observe also that the algebra $\mathcal{V}_n$ looks different from the above-mentioned Weyl-like algebras $B_{p,n}$ but it is isomorphic to the algebra $B_{n,n}$ and may be reduced to it by a change of the generators. We will return below to a geometric interpretation of the algebra $\mathcal{V}_n$.

Now we take the following determinant

$$D_0 = \begin{vmatrix} \theta_0(z_1) & \theta_1(z_1) & \dots & \theta_{n-2}(z_1) & \theta_{n-1}(z_1) \\ \theta_0(z_2) & \theta_1(z_2) & \dots & \theta_{n-2}(z_2) & \theta_{n-1}(z_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_0(z_n) & \theta_1(z_n) & \dots & \theta_{n-2}(z_n) & \theta_{n-1}(z_n) \end{vmatrix} =$$

$$c\exp(z_2+2z_3+\ldots+(n-1)z_n)\prod_{1\le i<j\le n}\theta(z_i-z_j)\theta(\sum_{i=1}^n z_i),$$

where the constant c is irrelevant for us because it will be cancelled in future computations.

Then we define the partial determinants D_i replacing the $i-$th column by the column of f_i, putting them on the place of the n-th column:

$$D_i=\begin{vmatrix}\theta_0(z_1) & \theta_1(z_1) & \ldots & \overset{i}{} & \theta_{n-1}(z_1) & f_1\\ \theta_0(z_2) & \theta_1(z_2) & \ldots & | & \theta_{n-1}(z_2) & f_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \theta_0(z_n) & \theta_1(z_n) & \ldots & | & \theta_{n-1}(z_n) & f_n\end{vmatrix}=$$

$$\sum_{1\le\alpha\le n}(-1)^{\alpha+n}\begin{vmatrix}\theta_0(z_1) & \theta_1(z_1) & \ldots & \theta_{n-1}(z_1) & \\ \theta_0(z_2) & \theta_1(z_2) & \ldots & \theta_{n-1}(z_2) & \\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \theta_0(z_n) & \theta_1(z_n) & \ldots & \theta_{n-1}(z_n) & \end{vmatrix}_{\alpha,i} f_\alpha.$$

Here the subscription $|\ |_{\alpha,i}$ means that we had to omit the i-th column and the α- th row.

The immediate corollary of the (1.1) is the following

Proposition 3.1. *The determinant ratios are formed a commutative family:*

$$[D_0^{-1}D_i, D_0^{-1}D_j]=0.$$

The result of the proposition can be expressed in an elegant way in the form of a commutation relation of generating functions.

Let us define a generating function $T(u)$ of a variable $u\in\mathbb{C}$:

$$T(u)=D_0^{-1}\sum_{1\le j\le n}(-1)^j\theta_j(u)D_j.$$

Then we can express the function $T(u)$, using the formulas for the determinants of the theta-functions as

$$T(u)=D_0^{-1}\sum_{1\le\alpha\le n}(-1)^\alpha\begin{vmatrix}\theta_0(z_1) & \theta_1(z_1) & \ldots & \theta_{n-1}(z_1)\\ \vdots & \vdots & \vdots & \vdots\\ \theta_0(u) & \theta_1(u) & \ldots & \theta_{n-1}(u)\\ \vdots & \vdots & \vdots & \vdots\\ \theta_0(z_n) & \theta_1(z_n) & \ldots & \theta_{n-1}(z_n)\end{vmatrix}f_\alpha= \tag{12}$$

$$=\sum_{1\le\alpha\le n}\frac{\theta(u+\sum\limits_{\beta\ne\alpha}z_\beta)\prod\limits_{1\le\beta\ne\alpha\le n}\theta(u-z_\beta)}{\prod\limits_{\beta\ne\alpha}\theta(z_\alpha-z_\beta)}\tilde f_\alpha,$$

where we denote by $\tilde{f}_\alpha$ the normalization

$$\tilde{f}_\alpha = \frac{f_\alpha}{\theta(\sum\limits_{i=1}^{n} z_i)}.$$

We remark that the commutation relations between the variables $z_1, \dots, z_n, \tilde{f}_1, \dots, \tilde{f}_n$ are the same as they were for the variables $z_1, \dots, z_n, f_1, \dots, f_n$.

In this notations the proposition now reads:

Proposition 3.2. *The "transfer-like" operators $T(u)$ commute for different values of the parameter u:*

$$[T(u), T(v)] = 0.$$

Now we will apply this result to a construction of the commuting elements in the algebra $Q_n(\mathcal{E}; \eta)$ for even n.

Let $n = 2m$. It is known in this case that the centre $Z(Q_{2m}(\mathcal{E}; \eta))$ for η of infinite order is generated by Casimir elements from $S^m(\Theta_{2m}(\Gamma))$ such that $f(z_1, \dots, z_m) = 0$ for $z_2 = z_1 + 2m\eta$. A straightforward computation shows that the space of such elements is two-dimensional and has as a basis the elements of the form

$$C_\alpha = \theta_\alpha(z_1 + \dots z_m) \prod_{1 \le i \ne j \le m} \theta(z_i - z_j - 2m\eta),$$

where $\theta_\alpha \in \Theta_2(\Gamma), \alpha \in \mathbb{Z}/2\mathbb{Z}$.

Let us fix an element $\Psi(z) \in \Theta_{m+5}(\Gamma)$ and two complex numbers $a, b \in \mathbb{C}$. Consider a family $f(u)(z_1, \dots, z_n)$ of elements from $S^m(\Theta_{2m}(\Gamma))$ such that

$$f(u)(z_1, z_1 + 2m\eta, z_2 \dots, z_{m-1})$$

$$= \Psi(z_1 + 4m^2\eta - \frac{1}{m+5}(a + (m-2)b + 2m(m-2)\eta))\theta(z_1 + \dots + z_{m-1} + a)\theta(z_1 + z_2 + b) \dots$$

$$\theta(z_1 + z_{m-1} + b)\theta(z_2 - z_1 - 4m\eta) \dots \theta(z_{m-1} - z_1 - 4m\eta)\theta(z_2 - z_1 + 2m\eta) \dots \theta(z_{m-1} - z_1 + 2m\eta) \times$$

$$\theta(u + z_2 + \dots + z_{m-1} - a - b + 2m\eta)\theta(u - z_2) \dots \theta(u - z_{m-1}) \times$$

$$\exp(2\pi i(2(m-2)z_1 + z_2 + \dots + z_{m-1})) \prod_{2 \le i \ne j \le m-1} \theta(z_i - z_j - 2m\eta).$$

The elements $f(u)$ are defined up to a linear combination of the Casimirs C_1, C_2 because of the annihilation of the Casimirs on the "diagonal" $z_1 = z_2 + 2m\eta$ (and we are defying the elements $f(u)$ namely on this "diagonal"!)

Theorem 3.1. *In the elliptic algebra $Q_{2m}(\mathcal{E}, \eta)$ the following relation holds*

$$[f(u), f(v)] = f(u) * f(v) - f(v) * f(u) = 0$$

Proof We will use the homomorphism $Q_{2m}(\mathcal{E},\eta) \to B_{m-1,2m}$ from the subsection 2.2.3. The element $f(u)$ is transposed by this homomorphism into the element

$$\sum_{1\le\alpha\le m-1} \frac{\theta(u+\sum\limits_{\beta\neq\alpha} z_\beta) \prod\limits_{1\le\beta\neq\alpha\le m-1} \theta(u-z_\beta)}{\prod\limits_{\beta\neq\alpha}\theta(z_\beta-z_\alpha)} f_\alpha,$$

where we had denote by the f_α the following expression

$$f_\alpha = \Psi(z_\alpha + 4m^2\eta - \frac{1}{m+5}(a+(m-2)b+2m(m-2)\eta))\times$$
$$\theta(\sum_{\beta=1}^{m-1} z_\beta + a)\prod_{\beta\neq\alpha}\theta(z_\alpha+z_\beta+b)\times$$
$$\exp(2\pi i(2(m-2)z_\alpha + \sum_{\beta\neq\alpha} z_\beta))e_\alpha e_1\dots e_{m-1}.$$

Then, the formula (12) and the proposition 3.2 give us immediately that the images of $f(u)$ under the homomorphism ϕ_{m-1} are commuting in $B_{m-1,2m}(\eta)$. It is known that the image of the algebra $Q_{2m}(\mathcal{E},\eta)$ in $B_{m-1,2m}(\eta)$ is the quotient of $Q_{2m}(\mathcal{E},\eta)$ over the centre $Z=\langle C_1,C_2\rangle$. Hence the commutator

$$[f(u),f(v)] = f(u)*f(v) - f(v)*f(u)$$

belongs to the ideal generated by the centre Z. To show that $[f(u),f(v)]=0$ we can consider the injective homomorphism into $B_{m,2m}(\eta)$ and it is sufficient to verify that the coefficient before $(e_1)^2\dots(e_m)^2$ equals to zero, or (which is equivalent) that

$$[f(u),f(v)](z_1,z_1+2m\eta,z_2,z_2+2m\eta,\dots,z_m,z_m+2m\eta)=0,$$

which is achieved easily by the direct verification.

Remark 3. A family of compatible quadratic Poisson structures such that its common linear combination is isomorphic to the Poisson structure of the classical elliptic algebra $q_n(\mathcal{E})$ was constructed in [31]. The Lenard scheme enables us with a family of Poisson commuting elements ("hamiltonians" in involution) in the algebra $q_n(\mathcal{E})$. These elements have the degree n if n is impair and $n/2$ otherwise.

Let $n=2m$. We conjecture that the constructed in 3.1 commuting elements in $Q_{2m}(\mathcal{E})$ are the quantum analogs of the commuting "hamiltonians" in $q_{2m}(\mathcal{E})$ generated by the Lenard scheme applied to the classical Casimirs ${C_0}^{(m)}, {C_1}^{(m)}$ of [31]. This conjecture is true in first non-trivial case $n=6$ when it is easily verified by direct computations.

3.2. **Commuting elements in "bosonization" of $Q_{n,n-1}(\mathcal{E},\eta)$.** The elliptic algebra $Q_{n,n-1}(\mathcal{E},\eta)$ is commutative and the homomorphism of "bosonization" provides a construction of a large class of the commuting families in the corresponding Weyl-like algebras. Some special choice of the relations between the numeric parameters (numbers of the generators) gives us in its turn the example of some new (to our knowledge) integrable system.

3.2.1. *Bosonization of the algebra $Q_{n,n-1}(\mathcal{E})$.*. The homomorphism $\phi_p : Q_n(\mathcal{E},\eta) \to B_{n,p}(\eta)$ from 2.2.3 may be extended and generalized to the case of the elliptic algebras $Q_{n,k}(\mathcal{E},\eta)$ ([27]). The structure of the Weyl-like algebra similar to $B_{n,p}(\eta)$ turns out to be more complicated for $k > 1$. We will use this extension in the special case $k = n-1$ i.e. for the commutative algebra $Q_{n,n-1}(\mathcal{E},\eta)$.

Let us consider the expansion of $\frac{n}{n-1}$ in continued fraction to obtain the numeric parameters of the dynamical Weyl-like algebra similar to $B_{n,p}(\eta)$:

$$\frac{n}{n-1} = 2 - \frac{1}{2 - \ldots - \frac{1}{2}}.$$

Let $\tilde{B}_{n,p_1,p_2,\ldots,p_{n-1}}(\eta)$ be an associative algebra generated by the following generators

$$z_{1,1},\ldots,z_{p_1,1}; e_{1,1},\ldots,e_{p_1,1};$$

$$z_{1,2},\ldots,z_{p_2,2}; e_{1,2},\ldots,e_{p_2,2}; \vdots$$

$$z_{1,n-1},\ldots,z_{p_{n-1},n-1}; e_{1,n-1},\ldots,e_{p_{n-1},n-1};$$

and the additional generators

$$t_{1,2}, t_{2,3}\ldots, t_{n-2,n-1};$$

$$f_{1,2}, f_{2,3}\ldots, f_{n-2,n-1};$$

We will impose the following commutation relations between them:

$$e_{\alpha,\gamma} z_{\beta,\gamma} = (z_{\beta,\gamma} - n\eta) e_{\alpha,\gamma}; \alpha \neq \beta$$

$$f_{\alpha,\alpha+1} t_{\alpha,\alpha+1} = (t_{\alpha,\alpha+1} - n\eta) f_{\alpha,\alpha+1}.$$

All other commutators we suppose to be equal to zero.

The commutative elliptic algebra $Q_{n,n-1}(\mathcal{E},\eta)$ admits the following "bosonization" homomorphism $\phi_{p_1,\ldots,p_{n-1}}$ which is a partial case of the general bosonization procedure for $Q_{n,k}(\mathcal{E},\eta)$.

More precisely,

$$\phi_{p_1,\ldots,p_{n-1}} : Q_{n,n-1}(\mathcal{E},\eta) \to \tilde{B}_{p_1,\ldots,p_{n-1}}(\eta)$$

can be squeezed through a homomorphism of the algebra $Q_{n-1,n}$ into an algebra of "exchange relations" with generators $e_{\alpha_1,\ldots,\alpha_{n-1}}$(see [30]). This "exchange" algebra, in its turn, can be imbed in the algebra $\tilde{B}_{p_1,\ldots,p_{n-1}}(\eta)$ with additional generators. This statement was established in the paper [28]. The explicit composition map $\phi_{p_1,\dots,p_{n-1}}$ can be expressed like the following "transfer-function"

$$\tilde{T}(u) = \sum_{\substack{1 \le \alpha_1 \le p_1 \\ \dots \\ 1 \le \alpha_{n-1} \le p_{n-1}}} \frac{\theta(u - z_{\alpha_1,1})\theta(u + z_{\alpha_1,1} - z_{\alpha_2,2})\theta(u + z_{\alpha_2,2} - z_{\alpha_3,3}) \dots \theta(u + z_{\alpha_{n-1},n-1})}{\prod\limits_{\substack{\beta \neq \alpha_\gamma \\ 1 \le \beta \le p_\gamma \\ 1 \le \gamma \le n-1}} \theta(z_{\alpha_\gamma,\gamma} - z_{\beta,\gamma})} \tag{13}$$

$$\theta(z_{\alpha_1,1} + z_{\alpha_2,2} - t_{1,2})\theta(z_{\alpha_2,2} + z_{\alpha_3,3} - t_{2,3}) \dots \theta(z_{\alpha_{n-2},n-2} + z_{\alpha_{n-1},n-1} - t_{n-2,n-1})$$
$$e_{\alpha_1,1} \dots e_{\alpha_{n-1,n-1}} f_{1,2} \dots f_{n-2,n-1}.$$

Then the following proposition is resulted from the definition of the combined homomorphism of the bosonization:

Proposition 3.3.
$$[\tilde{T}(u), \tilde{T}(v)] = 0.$$

We should observe that in the case of $p_1 = \dots = p_{n-1} = 2$ these commuting family gives an example of integrable system.

4. Some examples of elliptic integrable systems

4.1. **Low-dimensional example: Algebra** $q_2(\mathcal{E}, \eta)$**.** The elliptic integrable system arising in the 3.1 becomes very transparent in the case of $m = 1$ then the commutative elliptic algebra $Q_2(\mathcal{E}, \eta)$ has functional dimension 2 and its Poisson counterpart admits the Poisson morphism

$$\psi_2 : q_2(\mathcal{E}) \to b_2,$$

where the algebra b_2 are formed by the elements

$$\sum_{\alpha,\beta} f_{\alpha,\beta}(z_1, z_2) e_1^\alpha e_2^\beta,$$

where $f_{\alpha,\beta}(z_1, z_2)$ are meromorphic functions and the Poisson structure in b_2 is given by

$$\{z_i, z_j\} = \{e_i, e_j\} = \{e_i, z_i\} = 0 \; \{e_i, z_j\} = -2e_i (i \neq j), i, j = 1, 2. \tag{14}$$

The explicit formula for this mapping (for a given theta-function of the order 2 $f \in \Theta_2(\Gamma)$) is the following:

$$\psi_2(f) = \frac{f(z_1)}{\theta(z_1 - z_2)} e_1 + \frac{f(z_2)}{\theta(z_2 - z_1)} e_2. \tag{15}$$

Now, taking the basic theta-functions of the order 2 θ_1, θ_2, let us compute the Poisson brackets between their images $\psi_2(\theta_1)$ and $\psi_2(\theta_2)$:

Proposition 4.1. *These theta-functions commute in* b_2
$$\{\psi_2(\theta_1), \psi_2(\theta_2)\} = \psi_2(\{\theta_1, \theta_2\}) = 0.$$

4.2. **SOS eight-vertex model of Date-Miwa-Jimbo-Okado.** We remind in a neccessairy form some of the SOS eight-vertex model's ingredients (see [23]) and the construction of its transfer-operators to establish the relation between them and our operators $T(u)$.

This is a statistical mechanical model which is an *Interaction-round-a- face - model* (or IRF-model)-version of the Baxter model related to the elliptic quantum group of Felder $E_{\tau,\eta}(sl_2)$ which was studied by the Sklyanin's Separartion of Variables methods (under the antiperiodic boundary conditions) in [25] (see also [24] for the representation theory of $E_{\tau,\eta}(sl_2)$). We will use the results of [25] in the form which we need.

The antiperiodic boundary conditions of the model are fixed by the family of transfer-matrix $T_{SOS}(u,\lambda)$ where $u \in \mathbb{C}$ is a parameter and the family $T_{SOS}(u,\lambda)$ is expressed as a (twisted) traces of the $L-$operators of the model $L_{SOS}(u,\lambda)$ defined over so-called "auxiliary" space of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. The $L-$operator is built out in the tensor product of the fundamental representations of the elliptic quantum group and is twisted by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The $L-$ operator is usually represented in the form of $2\times 2-$-matrix of the form

$$L_{SOS}(u,\lambda) = \begin{pmatrix} a(u,\lambda) & b(u,\lambda) \\ c(u,\lambda) & d(u,\lambda) \end{pmatrix},$$

where the matrix entries are meromorphic on u and λ and obey the dynamical $RLL-$commutation relations for the Felder elliptic $R-$matrix

$$R(u,\lambda) = \begin{pmatrix} \theta(u+2\eta) & 0 & 0 & 0 \\ 0 & \frac{\theta(u)\theta(\lambda+2\eta)}{\theta(\lambda)} & \frac{\theta(u-\lambda)\theta(2\eta)}{\theta(\lambda)} & 0 \\ 0 & \frac{\theta(\lambda+u)\theta(2\eta)}{\theta(\lambda)} & \frac{\theta(u)\theta(\lambda-2\eta)}{\theta(\lambda)} & 0 \\ 0 & 0 & 0 & \theta(u+2\theta) \end{pmatrix}$$

(see *op.cit*).

We will have deal with the functional representations of $E_{\tau,\eta}(sl_2)$ which are the pairs (F,L) where F is a complex vector space of meromorphic functions $f(z_1,\dots,z_n,\lambda)$ (or a subspace of functions which are holomorphic in a part of the variables) and L is the $L-$operator as above. The entries of the $L-$operator are acting as the difference operators in the tensor product $V\otimes W$ where $dim V = 2$ such that the Felder $R-$matrix belongs to $\mathrm{End}(V\otimes V)$ and W is an appropriate subspace in the functional space F.

The Bethe ansatz method works for case of SOS model with periodic boundary conditions after Felder and Varchenko ([24]). In the antiperiodic case the family of the transfer matrices

$$T_{SOS}(u,\lambda) = trKL_{SOS}(u,\lambda),\ u \in \mathbb{C}$$

is commutative for the different parameter values: $[T_{SOS}(u,\lambda), T_{SOS}(v,\lambda)] = 0$, as it follows from the $RLL-$ relations by tediuos computations in [26].

In the other hand it is possible to establish the explicit one-to-one correspondence between the families of antiperiodic SOS transfer-matrices and the auxiliary transfer-matrices $T_{aux}(u,\lambda)$ (see 4.4.3 in [26]). This isomorphism is established by the version of the Separation of Variables.

The explicite expression of the auxiliar transfer-matrix is

$$T_{aux}(u,\lambda) = \sum_{\alpha=1}^{n} \frac{\theta(u+z_\alpha-\lambda)}{\theta(\lambda)} \prod_{1\le\beta\ne\alpha\le n} \frac{\theta(u+z_\beta)}{\theta(z_\beta-z_\alpha)} (\theta(z_\alpha+\eta)T_{z_\alpha}^{-2\eta} + \theta(z_\alpha-\eta)T_{z_\alpha}^{2\eta}),$$

where (to compare with our “transfer”-operators in the elliptic intgerable systems) we had put in the formulas of ([26], ch.4.6)

$$x_\alpha = 0,\ \Lambda_\alpha = 1,\ \alpha = 1, ..., n.$$

(The choice $\Lambda_\alpha = 1$ is corresponded to the case Separated Variables (prop. 4.36 in [26])).

The operators $T_{z_\alpha}^{\pm 2\eta}$ are acting in the way similar to the generators f_α above:

$$T_{z_\alpha}^{\pm 2\eta} f(z_1, ..., z_n) = f(z_1, ..., z_\alpha \pm 2\eta, ..., z_n) T_{z_\alpha}^{\pm 2\eta}.$$

Let us make the following change of the variables

$$f_\alpha^\pm = \theta(z_\alpha \mp \eta) T_{z_\alpha}^{\pm 2\eta}$$

Now the simple inspection of the formulas shows that

$$T_{aux}(u,\lambda) = T_+(u,\lambda) + T_-(u,\lambda),$$

and the the commutation results of 3.2 can be applied by the following

Proposition 4.2. *The ”transfer”-operator $T(u)$ coincides (up to an inessential numeric factor, depending on the normalization in the theta-function definition and rescaling of the parameter η) with the combinantion of the twisted traces of the auxiliary $L-$ operator of the antiperiodic SOS-model under the following choice of the variables:*

$$\lambda = \sum_{\alpha=1}^{n} z_\alpha,\ x_\alpha = 0,\ f_\alpha^\pm = \theta(z_\alpha \mp \eta) T_{z\alpha}^{\pm 2\eta}.$$

The proposition 3.2 gives a simple proof of the commutation relations for the twisted traces of the SOS-model in this special case. We should observe that the commutation relations

$$[T_+(u), T_-(v)] + [T_-(u), T_+(v)] = 0$$

are followed from the same arguments like in 3.2.

In other hand, we have got an additional argument to our belief that the "integrability" condition from [12] is equivalent in some sense to the another "integrability" form (in the given case to the $RLL-$relations).

We believe also that the role of the elliptic integrable systems in the IRF models does not restrict only to some specific conditions and we hope to return to this point in future.

4.3. **Elliptic analogs of the Beauville-Mukai systems and Fay identity.** Let us describe a geometric meaning of our IS's. We will start with an observation that the classical analog of the Weyl-like algebra $\mathcal{V}_n$ given by the Poisson brackets

$$0 = \{f_i, f_j\} = \{z_i, z_j\} = \{f_i, z_j\} \ (i \neq j), \ \{f_i, z_i\} = -nf_i$$

can be identified with the Poisson algebra of the functions on the symmetric product $S^n(Cone(\mathcal{E}))$ of a surface represented the elliptic curve cone (more precisely, on the Hilbert scheme $(Cone(\mathcal{E}))^{[n]}$ of the length n of the points on this surface.)

The n commuting elements $h_i = D_0^{-1}D_i$ in the Poisson algebra obtaining in the 3.2 can be interpreted as an elliptic version of the Beauville-Mukai systems associated with this Poisson surface.

We will develop in more details the first interesting case ($n = 3$) of the Poisson commuting conditions (the Plucker relations from sect.2) for this systems.

The Beauville-Mukai hamiltonians have the following form in this case:

$$H_1 = \frac{\det[e, \theta_1, \theta_2]}{\det[\theta_0, \theta_1, \theta_2]}, H_2 = \frac{\det[e, \theta_0, \theta_2]}{\det[\theta_0, \theta_1, \theta_2]}, H_3 = \frac{\det[e, \theta_0, \theta_1]}{\det[\theta_0, \theta_1, \theta_2]},$$

where the vectors-columns $e, \theta_i, \ i = 0, 1, 2$ have the following entries

$$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \theta_i = \begin{pmatrix} \theta_i(z_1) \\ \theta_i(z_2) \\ \theta_i(z_3) \end{pmatrix}, \ i = 0, 1, 2$$

Now the integrability condition (6) may be exprimed as a kind of 4-Riemann identity (known also as the trisecant Fay's identity):

$$\tilde{\theta}(v)\tilde{\theta}(u + \int_B^A \omega + \int_D^C)\omega = \tilde{\theta}(u + \int_D^A \omega)\tilde{\theta}(u + \int_B^C \omega)\frac{E(A,B)E(D,C)}{E(A,C)E(D,B)} + \tag{16}$$
$$\tilde{\theta}(u + \int_B^A \omega)\tilde{\theta}(u + \int_D^C \omega)\frac{E(A,D)E(C,B)}{E(A,C)E(D,B)},$$

where $\int_X^Y : \mathcal{E} \otimes \mathcal{E} \to \mathrm{Jac}(\mathcal{E}) = \mathcal{E}$ is a Jacobi mapping $X, Y \in \mathcal{E}, \int_X^Y := \int_X^Y \omega$ for a chosen holomorphic differential ω on the elliptic curve,
$E(X,Y) = -E(Y,X)$ is a prime form:

$$\frac{E(A,B)E(D,C)}{E(A,C)E(D,B)} = \frac{\tilde{\theta}(\int_A^B)\tilde{\theta}(\int_D^C)}{\tilde{\theta}(\int_A^C)\tilde{\theta}(\int_D^B)}$$

and $\tilde{\theta}(u)$ is an odd theta-function. (see [45].)

Let a, b, c, d be four points corresponded to the point A, B, C, D in (16) and we will choose $u = a + c$.

Then the (16) reads as

$$\tilde{\theta}(a+c)\tilde{\theta}(a-c)\tilde{\theta}(b+d)\tilde{\theta}(b-d) - \tilde{\theta}(a+b)\tilde{\theta}(a-b)\tilde{\theta}(c+d)\tilde{\theta}(c-d) + \tilde{\theta}(a+d)\tilde{\theta}(a-d)\tilde{\theta}(c+b)\tilde{\theta}(c-b) = 0, \tag{17}$$

where we are easily recognizing a form of the commutativity conditions (6) for the case $n = 3$ and an appropriate choice of a linear relation between 4 points a, b, c, d and $z_1, z_2, z_3, \eta/3$ (modulo an irrelevant exponential factor entering in the relations between the theta-functions in different normalizations).

Remark 4. 1. The appearance of the Riemann-Fay relations as a commutativity or an integrability conditions in this context looks quite natural both from the "Poissonian " as well as from the "integrable" viewpoints. The "Poisson-Plucker" relations and their generalizations in the context of the Poisson polynomial structures were studied in [44]. In the other hand, the first links between the integrability conditions (in the form of Hirota bilinear identities for some elliptic difference many-body-like systems) and the Fay formulas were established in [17].

2. Second remark concerns the relations between the Fay trisecant formulas on an elliptic curve and a version of a "triangle" relations known as the "associative" Yang-Baxter equation obtained by Polischuk ([42]). This result gives an additional evidence that the commutation relations (2),(3) could be interpreted as an algebraic sort of Yang-Baxter equation. An amusing appearance of the NC determinants in both constructions (the Cartier-Foata determinants defined above are of course a partial case of the quasi-determinants of Gelfand-Retakh [7]) shows that the ideas of [12] might be useful in "non-commutative integrability" constructions which involve quasi-determinants, quasi-Plucker relations etc.

5. Discussion and future problems

We have proved that the Elliptic Algebras (under some mild restrictions) carry the families of commuting elements which become in some cases the examples of Integrable Systems.

Let us indicate some questions deserving future investigations.

We are going to construct an analogs of the IS on the algebras $Q_{2m}(\mathcal{E}, \eta)$ which are corresponded to the maximal symplectic leaves of the Elliptic Algebras $Q_{2m+1}(\mathcal{E}, \eta)$ using the bi-hamiltonian elliptic families. This analog should quantize the bi-hamiltonian families in $\mathbb{C}^n$ in the case of an impair n.

One of the main motivations to construct the systems on $\mathcal{V}_n$ explicitly in the terms of the determinants of $\theta-$functions of order n was the tentative to find a confirmation to the hypothetical "separated" form of the hamiltoninans to $n-$point Double-Elliptic system, proposed in the paper [4].

The relevance of our construction to this circle of the problems had got an additional evidence in the paper [35]. Recent discussions around of the integrability in Dijkgraaf-Vafa and Seiberg-Witten theory provide us with the "physical" insights to support the relation between the Beauville-Mukai and Double Elliptic IS (see [6], [5]).

The future paper (in collaboration with A. Gorsky) which should clarify the place of the IS associated with the Elliptic Algebras and the integrability phenomena in SUSY gauge theories are in progress.

There are some open questions about the relations of these IS with the Beauville-Mukai Lagrangian fibrations on the Hilbert scheme $(\mathbb{P}^2\backslash\mathcal{E})^{[n]}$ as well as with their non-commutative counterparts proposed in [43]. We can argue that the proposed in [9] quantization scheme for the fraction fields can be applied to the NC surface $(\mathbb{P}_S\backslash\mathcal{E})$ and to "extend" the deformation to the NC Hilbert scheme $(\mathbb{P}_S\backslash\mathcal{E})^{[n]}$ introducing in [43]. The proposition 3.1 from [12] then should in principle to give an NC integrable Beauville-Mukai system on $(\mathbb{P}_S\backslash\mathcal{E})^{[n]}$ as a NC IS associated with the Cherednik algebras of [10]. These questions are also in the scope of our interest.

Finally, we should mention an interesting question of generalization of the Beauville-Mukai IS which was introduced by Gelfand-Zakharevich on the Hilbert scheme $(X_9)^{[n]}$ of the Del Pezzo surface X_9 getting by the blow-up of 9 points on $\mathbb{P}^2$. These 9 points are the intersection points of two cubic plane curves. Let π is a Poisson tensor on X_9 then it can be extend to the whole $\mathbb{P}^2$ such the extension $\tilde{\pi}$ has 9 zeroes in these 9 points. The polynomial degree of the tensor $\tilde{\pi}$ is 3 and it follows that this polynomial is a linear combination of the polynomials corresponding to the initial elliptic curves. This bi-hamiltonian structure corresponds to the case $n = 3$ studied in the paper [31]. The open question arises - what an algebro-geometric construction behind the bi-hamiltonian Elliptic Poisson Algebra of [31] in the case of arbitrary n?

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