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Noncommutative Toda chains, Hankel quasideterminants and the Painlevé II equation

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Abstract
We construct solutions of an infinite Toda system and an analog of the Painlevé II equation over noncommutative differential division rings in terms of quasideterminants of Hankel matrices.

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Introduction

Let R be an associative algebra over a field with a derivation D. Set $Df = f'$ for any $f \in R$. Assume that R is a division ring. In this paper we construct solutions for the system of equations (0.1) over algebra R

$$\left(\theta_n'\theta_n^{-1}\right)' = \theta_{n+1}\theta_n^{-1} - \theta_n\theta_{n-1}^{-1}, \qquad n \geqslant 1 \qquad (0.1-n)$$

assuming that $\theta_1 = \phi, \theta_0 = \psi^{-1}, \phi, \psi \in R$ and its 'negative' counterpart (0.1′)

$$\left(\eta_{-m}^{-1}\eta_{-m}'\right)' = \eta_{-m}^{-1}\eta_{-m-1} - \eta_{-m+1}^{-1}\eta_{-m}, \qquad m \geqslant 1 \qquad (0.1'-m)$$

where $\eta_0 = \phi^{-1}, \eta_{-1} = \psi$.

Note that $\theta'\theta^{-1}$ and $\theta^{-1}\theta'$ are noncommutative analogs of the logarithmic derivative $(\log\theta)'$.

We then use the solutions of the Toda equations under a certain ansatz for constructing solutions of the *noncommutative Painlevé II equation*

$$P_{II}(u,\beta):\quad u'' = 2u^3 - 2xu - 2ux + 4\left(\beta + \frac{1}{2}\right),$$

where $u, x \in R$, $x' = 1$ and β is a scalar parameter, $\beta' = 0$.

Unlike papers [NGR] and [N] we consider here a 'pure noncommutative' version of the Painlevé equation without any additional assumption for our algebra R.

In fact a noncommutative ('matrix') version of Painlevé II

$$P_{II}(u, \beta): \quad u'' = 2u^3 + xu + \beta I$$

was considered for the first time in the papers by Sokolov with different co-authors: we mention here for example [BS]. But their form of this equation, satisfying the Painlevé test, at the same time cannot be obtained as a reduction of some matrix analog of the mKDV system.

Our equation is similar to this noncommutative Painlevé II but there is an essential difference: we write the second term on the RHS in the symmetric or 'anticommutator' form. This splitting form is much more adaptable to some generalizations of the usual commutative Painlevé II.

Our motivation is the following. In the commutative case one can consider an infinite Toda system (see for example [KMNOY, JKM]):

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \phi\psi\tau_n^2 \qquad (0.2-n)$$

with the conditions $\tau_1 = \phi, \tau_0 = 1, \tau_{-1} = \psi$.

Let $n \geqslant 1$. By setting $\theta_n = \tau_n/\tau_{n-1}$ the system can be written as

$$(\log\tau_n)'' = \theta_{n+1}\theta_n^{-1} - \phi\psi.$$

For $n = 1$ we have equation (0.1-1) with $\theta_1 = \phi, \theta_0 = \psi^{-1}$. By subtracting equation (2.2-n) from (2.2-(n+1)) and replacing the difference $\log\tau_{n+1} - \log\tau_n$ by $\log\frac{\tau_{n+1}}{\tau_n}$ one can get (0.1–n).

Similarly, the system (0.2-m) for positive m implies the system (0.1′–m) for $\theta_{-m} = \tau_{-m}/\tau_{-m+1}$.

By going from τ_n's to their consecutive relations we are cutting the system of equations parametrized by $-\infty < n < \infty$ to its 'positive' and 'negative' part.

A special case of the semi-infinite system (0.1) over noncommutative algebra with θ_0^{-1} formally equal to zero was treated in [GR2]. In this paper, solutions of the Toda system (0.1) with $\theta_0^{-1} = 0$ were constructed as *quasideterminants* of certain Hankel matrices. It was the first application of quasideterminants introduced in [GR1] to noncommutative integrable systems. This line was continued by several researchers, see, for example, [EGR2, EGR1], papers by Glasgow school [GNS, GN, GNO] and a recent paper [DFK].

In this paper we generalize the result of [GR2] for $\theta_0 = \psi^{-1}$ and extend it to the infinite Toda system. The solutions are also given in terms of quasideterminants of Hankel matrices but the computations are much harder. We follow here the commutative approach developed in [KMNOY, JKM] with some adjustments but our proofs are far from a straightforward generalization. In particular, for our proof we have to introduce and investigate *almost Hankel matrices* (see section 2.2).

From solutions of the systems (0.1) and (0.1′) under certain ansatz we deduce solutions for the noncommutative equation $P_{II}(u, \beta)$ for various parameters β (theorem 3.2). This is a noncommutative development of an idea from [KM].

We start this paper by a reminder of the basic properties of quasideterminants, then construct solutions of the systems (0.1) and (0.1′) and then apply our results to noncommutative Painlevé II equations following the approach by [KM].

Our paper shows that a theory of 'pure' noncommutative Painlevé equations and the related τ-functions can be rather rich and interesting. The Painlevé II type was chosen as a model and we will investigate other types of Painlevé equations.

1. Quasideterminants

The notion of quasideterminants was introduced in [GR1], see also [GGRW, GR2, GR3].

Let $A = ||a_{ij}||$, $i, j = 1, 2, \dots, n$, be a matrix over an associative unital ring. Denote by A^{pq} the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the pth row and the qth column. Let r_i be the row matrix $(a_{i1}, a_{i2}, \dots \hat{a}_{ij}, \dots, a_{in})$ and c_j be the column matrix with entries $(a_{1j}, a_{2j}, \dots \hat{a}_{ij}, \dots, a_{nj})$.

For $n = 1$, $|A|_{11} = a_{11}$. For $n > 1$ the *quasideterminant* $|A|_{ij}$ is defined if the matrix A^{ij} is invertible. In this case

$$|A|_{ij} = a_{ij} - r_i (A^{ij})^{-1} c_j.$$

If the inverse matrix $A^{-1} = ||b_{pq}||$ exists, then $b_{pq} = |A|_{qp}^{-1}$ provided that the quasideterminant is invertible.

If R is commutative, then $|A|_{ij} = (-1)^{i+j} \det A / \det A^{ij}$ for any i and j.

Examples.

(a) For the generic (2×2)-matrix $A = (a_{ij})$, $i, j = 1, 2$, there are four quasideterminants:

$$|A|_{11} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \quad |A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22},$$
$$|A|_{21} = a_{21} - a_{22} a_{12}^{-1} a_{11}, \quad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}.$$

(b) For the generic (3×3)-matrix $A = (a_{ij})$, $i, j = 1, 2, 3$, there are nine quasideterminants. One of them is

$$|A|_{11} = a_{11} - a_{12}\left(a_{22} - a_{23} a_{33}^{-1} a_{32}\right)^{-1} a_{21} - a_{12}\left(a_{32} - a_{33} a_{23}^{-1} a_{22}\right)^{-1} a_{31}$$
$$- a_{13}\left(a_{23} - a_{22} a_{32}^{-1} a_{33}\right)^{-1} a_{21} - a_{13}\left(a_{33} - a_{32} \cdot a_{22}^{-1} a_{23}\right)^{-1} a_{31}.$$

Here are the transformation properties of quasideterminants. Let $A = ||a_{ij}||$ be a square matrix of order n over a ring R.

(i) The quasideterminant $|A|_{pq}$ does not depend on permutations of rows and columns in the matrix A that do not involve the pth row and the qth column.

(ii) *The multiplication of rows and columns.* Let the matrix $B = ||b_{ij}||$ be obtained from the matrix A by multiplying the ith row by $\lambda \in R$ from the left, i.e. $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for $k \neq i$. Then

$$|B|_{kj} = \begin{cases} \lambda |A|_{ij} & \text{if} \quad k = i, \\ |A|_{kj} & \text{if} \quad k \neq i \end{cases} \qquad \text{and} \qquad \lambda \text{ is invertible.}$$

Let the matrix $C = ||c_{ij}||$ be obtained from the matrix A by multiplying the jth column by $\mu \in R$ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all i and $l \neq j$. Then

$$|C|_{i\ell} = \begin{cases} |A|_{ij}\mu & \text{if} \quad l = j, \\ |A|_{i\ell} & \text{if} \quad l \neq j \end{cases} \qquad \text{and} \qquad \mu \text{ is invertible.}$$

(iii) *The addition of rows and columns.* Let the matrix B be obtained from A by replacing the kth row of A with the sum of the kth and lth rows, i.e. $b_{kj} = a_{kj} + a_{lj}$, $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$|A|_{ij} = |B|_{ij}, \qquad i = 1, \dots k-1, \quad k+1, \dots n, \qquad j = 1, \dots, n.$$

We will sometimes need the following property of quasideterminants called the *noncommutative Lewis Carroll identity*. It is a special case of the *noncommutative Sylvester identity* from [GR1, GR2] or *heredity principle* formulated in [GR3].

Let $A = ||a_{ij}||$, $i, j = 1, 2, \dots, n$. Consider the following $(n-1) \times (n-1)$-submatrices $X = ||x_{pq}||$, $p, q = 1, 2, \dots, n-1$, of A: matrix $A_0 = ||a_{pq}||$ obtained from A by deleting its

nth row and nth column; matrix $B = ||b_{pq}||$ obtained from A by deleting its $(n-1)$st row and nth column; matrix $C = ||c_{pq}||$ obtained from A by deleting its nth row and $(n-1)$st column; matrix $D = ||d_{pq}||$ obtained from A by deleting its $(n-1)$st row and $(n-1)$st column. Then

$$|A|_{nn} = |D|_{n-1,n-1} - |B|_{n-1,n-1}|A_0|^{-1}_{n-1,n-1}|C|_{n-1,n-1}. \tag{1.1}$$

2. Quasideterminant solutions of noncommutative Toda equations

2.1. Noncommutative Toda equations in bilinear form

Let F be a commutative field and R be an associative ring containing F-algebra. Let $D : R \to R$ be a derivation over F, i.e. an F-linear map satisfying the Leibniz rule $D(ab) = D(a) \cdot b + a \cdot D(b)$ for any $a, b \in R$. Also, $D(\alpha) = 0$ for any $\alpha \in F$. As usual, we set $u' = D(u), u'' = D(D(u)), \ldots$. Recall that $D(v^{-1}) = -v^{-1}v'v^{-1}$ for any invertible $v \in R$.

Let $\phi, \psi \in R$ and R be a division ring. We now construct solutions for the noncommutative Toda equations (0.1) and (0.1′) assuming that $\theta_0 = \psi^{-1}, \theta_1 = \phi$ and $\eta_0 = \phi^{-1}, \eta_{-1} = \psi$.

Set (cf [KMNOY, JKM] for the commutative case) $a_0 = \phi, b_0 = \psi$ and

$$a_n = a'_{n-1} + \sum_{i+j=n-2, i,j\geqslant 0} a_i\psi a_j, \qquad b_n = b'_{n-1} + \sum_{i+j=n-2, i,j\geqslant 0} b_i\phi b_j, \qquad n \geqslant 1. \tag{2.1}$$

Construct Hankel matrices $A_n = ||a_{i+j}||, B_n = ||b_{i+j}||, i, j = 0, 1, 2 \ldots, n$.

Theorem 2.1. *Set $\theta_{p+1} = |A_p|_{p,p}$, $\eta_{-q-1} = |B_q|_{q,q}$. The elements θ_n for $n \geqslant 1$ satisfy the system (0.1) and the elements η_{-m}, $m \geqslant 1$, satisfy the system (0.1′).*

This theorem can be viewed as a noncommutative generalization of theorem 2.1 from [KMNOY]. In [KMNOY] it was proved that in the commutative case the Hankel determinants $\tau_{n+1} = \det A_n, n \geqslant 0, \tau_0 = 1, \tau_{-n-1} = \det B_n, n \leqslant 0$, satisfy the system (0.2).

Example. The (noncommutative) logarithmic derivative $\theta'_1\theta_1^{-1}$ satisfies the noncommutative Toda equation (0.1-1):

$$\left(\theta'_1\theta_1^{-1}\right)' = \theta_2\theta_1^{-1} - \phi\psi.$$

In fact,

$$\left(\theta'_1\theta_1^{-1}\right)' = \left(a_1a_0^{-1}\right)' = (a_2 - a_0\psi a_0)a_0^{-1} - \left(a_1a_0^{-1}\right)^2$$
$$= \left(a_2 - a_1a_0^{-1}a_1\right)a_0^{-1} - a_0\psi = \theta_2\theta_1^{-1} - \phi\psi.$$

Our proof of theorem 2.1 in the general case is based on the properties of quasideterminants of almost Hankel matrices.

2.2. Almost Hankel matrices and their quasideterminants

We define *almost Hankel* matrices $H_n(i, j) = ||a_{st}||$, $s, t = 0, 1, \ldots, n$, $i, j \geqslant 0$, for a sequence $a_0, a_1, a_2, \ldots$ as follows. Set $a_{nn} = a_{i+j}$ and for $s, t < n$

$$a_{s,t} = a_{s+t}, \qquad a_{n,t} = a_{i+t}, \qquad a_{s,n} = a_{s+j}$$

and $a_{nn} = a_{i+j}$.

Note that $H_n(n, n)$ is a Hankel matrix.

Denote by $h_n(i, j)$ the quasideterminant $|H_n(i, j)|_{nn}$. Then $h_n(i, j) = 0$ if at least one of the inequalities $i < n$, $j < n$ holds.

Lemma 2.2.

$$h_n(i,j)' = \kappa_n(i,j) - \sum_{p=1}^{i} a_{p-1}\psi h_n(i-p,j) - \sum_{q=1}^{j} h_n(i,j-q)\psi a_{q-1} \quad (2.2)$$

where

$$\kappa_n(i,j) = h_n(i+1,j) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j). \quad (2.3a)$$

Also,

$$= h_n(i,j+1) - h_n(i,n)h_{n-1}^{-1}(n-1,n-1)h_{n-1}(n-1,j). \quad (2.3b)$$

Note that some summands $h_n(i-p,j)$, $h_n(i,j-q)$ in formula (2.2) can be equal to zero. Since $h_n(i,j) = 0$ when $i < n$ or $j < n$ we have the following corollary.

Corollary 2.3.

$$h_n(n,n)' = \kappa_n(n,n),$$

$$h_n(i,n)' = \kappa_n(i,n) - \sum_{s=1}^{i} a_{s-1}\psi h_n(i-s,n),$$

$$h_n(n,j)' = \kappa_n(n,j) - \sum_{v=1}^{j} h_n(n,j-v)\psi a_{v-1}.$$

Proof of lemma 2.2. We prove lemma 2.2 by induction. By definition,

$$h_1(i,j)' = a_{i+j+1} - \sum_{k=0}^{i+j-1} a_k\psi a_{i+j-1-k} - \left(a_{i+1} - \sum_{s=0}^{i-1} a_s\psi a_{i-1-s}\right)a_0^{-1}a_j$$
$$+ a_i a_0^{-1}a_1 a_0^{-1}a_j - a_i a_0^{-1}\left(a_{j+1} - \sum_{t=0}^{j-1} a_{j-1-t}\psi a_t\right).$$

Set

$$\kappa_1(i,j) = a_{i+j+1} - a_{i+1}a_0^{-1}a_j + a_i a_0^{-1}a_1 a_0^{-1}a_j - a_i a_0^{-1}a_{j+1};$$

we can check formulas (2.3*a*) and (2.3*b*). The rest of the proof for $n = 1$ is easy.

Assume now that formula (2.2) is true for $n \geqslant 1$ and prove it for $n + 1$. By the noncommutative Sylvester identity (1.1)

$$h_{n+1}(i,j) = h_n(i,j) - h_n(i,n)h_n^{-1}(n,n)h_n(n,j). \quad (2.4)$$

Set $h_{n+1}(i,j)' = \kappa_{n+1}(i,j) + r_{n+1}(i,j)$ where κ_{n+1} contains all terms without ψ. Then

$$\kappa_{n+1}(i,j) = \kappa_n(i,j) - \kappa_n(i,n)h_n^{-1}(n,n)h_n(n,j)$$
$$+ h_n(i,n)h_n^{-1}(n,n)\kappa_n(n,n)h_n^{-1}(n,n)h_n(n,j) - h_n(i,n)h_n^{-1}(n,n)\kappa_n(n,j).$$

By induction, the first two terms can be written as

$$h_n(i+1,j) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j)$$
$$+ [h_n(i+1,n) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,n)]h_n^{-1}(n,n)h_n(n,j)$$
$$= h_n(i+1,j) - h_n(i+1,n)h_n^{-1}(n,n)h_n(n,j).$$

This expression equals to $h_{n+1}(i+1,j)$ by the Sylvester identity.

The last two terms in $\kappa_{n+1}(i, j)$ can be written as

$$
\begin{aligned}
h_n(i,n)h_n^{-1}(n,n)[h_n(n+1,n) - h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,n)]h_n^{-1} \\
\times (n,n)h_n(n,j) - h_n(i,n)h_n^{-1}(n,n)[h_n(n+1,j) \\
- h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j)] \\
= h_n(i,n)h_n^{-1}(n,n)[-h_n(n+1,n) + h_n(i,n)h_n^{-1}(n,n)h_n(n+1,j)] \\
= -h_n(i,n)h_n^{-1}(n,n)h_{n+1}(n+1,j)
\end{aligned}
$$

also by the Sylvester identity.

Therefore, $\kappa_{n+1}(i, j)$ satisfies formula (2.3*a*). Formula (2.3*b*) can be obtained in a similar way.

Let us look at the terms containing ψ. According to the inductive assumption

$$h_n(i,j)' = \kappa_n(i,j) - \sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,j) - \sum_{\ell=1}^{j} h_n(i,j-\ell)\psi a_{\ell-1}.$$

Using corollary 2.3 and formula (2.2) for n one can write $r_{n+1}(i, j)$ as

$$
\begin{aligned}
-\sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,j) - \sum_{\ell=1}^{j} h_n(i,j-\ell)\psi a_{\ell-1} \\
+ \sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,n)]h_n^{-1}(n,n)h_n(n,j) \\
+ h_n(i,n)h_n^{-1}(n,n)\sum_{\ell=1}^{j} h_n(n,j-\ell)\psi a_{\ell-1} \\
= -\sum_{k=1}^{i} a_{k-1}\psi[h_n(i-k,j) - h_n(i-k,n)h_n^{-1}(n,n)h_n(n,j)] \\
- \sum_{\ell=1}^{j}[h_n(i,j-\ell) - h_n(i,n)h_n^{-1}(n,n)h_n(n,j-\ell)]\psi a_{\ell-1}.
\end{aligned}
$$

Our lemma now follows from the Sylvester identity applied to each expression in square brackets. □

Corollary 2.3 and formula (2.3*a*) immediately imply

Corollary 2.4. *For* $n > 1$

$$h_n(n,n)'h_n^{-1}(n,n) = h_n(n+1,n)h_n^{-1}(n,n) - h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1).$$

Note on the right-hand side we have a difference of left quasi-Plücker coordinates (see [GR3]).

2.3. Proof of theorem 2.1

Our solution of the Toda system (0.1) follows from corollary 2.4 and the following lemma.

Lemma 2.5. *For* $k > 0$

$$[h_k(k+1,k)h_k^{-1}(k,k)]' = h_{k+1}(k+1,k+1)h_k^{-1}(k,k) - a_0\psi.$$

Proof. Corollary 2.3 and formula (2.3*b*) imply

$$h_k(k+1,k)' = h_k(k+1,k+1) - h_k(k+1,k)h_{k-1}^{-1}(k-1,k-1)h_{k-1}(k-1,k) - a_0\psi h_k(k,k)$$

because $h_k(k+1-s,k) = 0$ for $s > 1$.

Then, using again formula (2.3*b*) one has

$$\begin{aligned}
[h_k(k+1,k)'h_k^{-1}(k,k)]' &= [h_k(k+1,k+1) - h_k(k+1,k)h_{k-1}^{-1} \\
&\quad \times (k-1,k-1)h_{k-1}(k-1,k) - a_0\psi h_k(k,k)]h_k^{-1}(k,k) \\
&\quad - h_k(k+1,k)h_k^{-1}(k,k)][h_k(k,k+1) - h_k(k,k)h_{k-1}^{-1} \\
&\quad \times (k-1,k-1)h_{k-1}(k-1,k)]h_k^{-1}(k,k) \\
&= [h_k(k+1,k+1) - h_k(k+1,k)h_k^{-1}(k,k)h_k(k,k+1)]h_k^{-1}(k,k) - a_0\psi \\
&= h_{k+1}(k+1,k+1)h_k^{-1}(k,k) - a_0\psi
\end{aligned}$$

by the Sylvester formula. □

Theorem 2.1 now follows from corollary 2.4 and lemma 2.5. The statement for η_{-m}, $m \geqslant 1$ can be proved in a similar way.

3. Noncommutative Painlevè II

3.1. Commutative Painlevè II and Hankel determinants: motivation

The Painlevè II (P_{II}) equation (with commutative variables)

$$u'' = 2u^3 - 4xu + 4\left(\beta + \frac{1}{2}\right)$$

admits unique rational solution for a half-integer value of the parameter β. These solutions can be expressed in terms of logarithmic derivatives of ratios of Hankel-type determinants. Namely, if $\beta = N + \frac{1}{2}$, then

$$u = \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{\det A_{N+1}(x)}{\det A_N(x)},$$

where $A_N(x) = ||a_{i+j}||$ where $i, j = 0, 1, \ldots, n-1$. The entries of the matrix are polynomials $a_n(x)$ subjected to the recurrence relations:

$$a_0 = x, \quad a_1 = 1, \quad a_n = a'_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-2-i}$$

(see [KO], [JKM]).

3.2. Noncommutative and 'quantum' Painlevè II

We will consider here a *noncommutative* version of P_{II} which we will denote nc $- P_{II}(x,\beta)$:

$$u'' = 2u^3 - 2xu - 2ux + 4\left(\beta + \frac{1}{2}\right),$$

where $x, u \in R$, $\quad x' = 1$ and β is a central scalar parameter ($\beta \in F$, $\beta' = 0$).

This equation is a specialization of a general noncommutative Painlevé II system with respect to three dependent noncommutative variables u_0, u_1, u_2:

$$\begin{aligned}
u'_0 &= u_0u_2 + u_2u_0 + \alpha_0 \\
u'_1 &= -u_1u_2 - u_2u_1 + \alpha_1 \\
u'_2 &= u_1 - u_0.
\end{aligned}$$

Indeed, taking the derivative of the third and using the first and second, we get

$$u_2'' = -(u_0+u_1)u_2 - u_2(u_0+u_1) + \alpha_1 - \alpha_0.$$

Then we have

$$(u_0+u_1)' = -u_2'u_2 - u_2u_2' + \alpha_0 + \alpha_1$$

and immediately

$$-(u_0+u_1) = u_2^2 - (\alpha_0+\alpha_1)x - \gamma, \quad \gamma \in F.$$

Comparing with u_2'' we obtain the following $\mathrm{nc} - P_{II}$:

$$u_2'' = 2u_2^3 - (\alpha_0+\alpha_1)xu_2 - (\alpha_0+\alpha_1)u_2x - 2\gamma u_2 + \alpha_1 - \alpha_0.$$

Our equation corresponds the choice $\gamma = 0, \alpha_1 = 2(\beta+1), \alpha_0 = -2\beta$.

Remark. The noncommutative Painlevé II system above is the straightforward generalization of the analog system in [NGR] when the variables $u_i, i = 0, 1, 2$, are subordinated to some commutation relations. Here we do not assume that the 'independent' variable x commutes with u_i.

Going further with this analogy we will write a 'fully noncommutative' Hamiltonian of the system

$$H = \tfrac{1}{2}(u_0u_1 + u_1u_0) + \alpha_1u_2$$

and introduce the 'canonical' variables

$$p := u_2, \qquad q := u_1, \qquad x := \tfrac{1}{2}\big(u_0 + u_1 + u_2^2\big).$$

Proposition 3.1. *Let a triple (x, p, q) be a 'solution' of the 'Hamiltonian system' with the Hamiltonian H and $\alpha_1 = 2(\beta+1)$:*

$$p_x = -H_q$$
$$q_x = H_p.$$

Then p satisfies the $\mathrm{nc} - P_{II}$:

$$p_{xx} = 2p^3 - 2px - 2xp + 4\left(\beta + \frac{1}{2}\right).$$

Proof. Straightforward computation gives that

$$p_x = p^2 + 2q - 2x$$
$$q_x = \alpha_1 - (qp + pq).$$

Taking $p_{xx} = p_xp + pp_x + 2q_x - 2$ and substituting p_x and q_x we obtain the result. □

We give (for the sake of completeness) the explicit expression of the Painlevé Hamiltonian H in the 'canonical' coordinates:

$$H(x, p, q) = qx + xq - q^2 - \tfrac{1}{2}(qp^2 + p^2q) + 2(\beta+1)p.$$

3.3. Solutions of the noncommutative Painlevé and the Toda system

Theorem 3.2. *Let ϕ and ψ satisfy the following identities:*

$$\psi^{-1}\psi'' = \phi''\phi^{-1} = 2x - 2\phi\psi, \tag{3.1}$$

$$\psi\phi' - \psi'\phi = 2\beta. \tag{3.2}$$

Then for $n \in \mathbb{N}$

(1) $u_n = \theta_n'\theta_n^{-1}$ satisfies $nc - P_{II}(x, \beta + n - 1)$;
(2) $u_{-n} = \eta_{-n}'\eta_{-n}^{-1}$ satisfies $nc - P_{II}(x, \beta - n)$.

Let us start with the following useful (though slightly technical) lemma.

Lemma 3.3. *Under the conditions of theorem 2.1 we have the chain of identities ($n \geqslant 0$):*

(1) $\theta_n'\theta_n^{-1} + \theta_{n-1}'\theta_{n-1}^{-1} = 2(\beta + n - 1)\theta_{n-1}\theta_n^{-1}$
(2) $\theta_n''\theta_n^{-1} = 2\big(x - \theta_n\theta_{n-1}^{-1}\big)$
and also, for $n \geqslant 1$
(3) $\eta_{-n}^{-1}\eta_{-n}' + \eta_{-n+1}\eta_{-n+1}' = -2(\beta - n + 1)\eta_{-n}^{-1}\eta_{-n}^{-1}\eta_{-n+1}$
(4) $\eta_{-n}^{-1}\eta_{-n}'' = 2(x - \eta_{-n+1}^{-1}\eta_{-n}$.

Proof. Remark that the first step in the chain ($n = 1$) directly follows from our assumption: $\theta_1 = \phi, \theta_0 = \psi^{-1}$:

$$\phi'\phi^{-1} + (\psi^{-1})'\psi = 2\beta\psi^{-1}\phi^{-1}.$$

Indeed, we have

$$\phi'\phi^{-1} - \psi^{-1}\psi' = 2\beta\psi^{-1}\phi^{-1},$$

where the result

$$\psi\phi' - \psi'\phi = 2\beta.$$

The second step ($n = 2$) is a little bit tricky.

We consider the Toda equation $(\phi'\phi^{-1})' = \theta_2\phi^{-1} - \phi\psi$ and easily find θ_2 (using $\phi''\phi^{-1} = 2x - 2\phi\psi$):

$$\theta_2 = 2x\phi - \phi\psi\phi - (\phi'\phi^{-1})\phi'.$$

Taking the derivation and using the same Toda and the first step identity, we get

$$\theta_2' = 2\phi(\beta + 1) - \phi'\phi^{-1}\theta_2.$$

The second ($n = 2$) identity is rather straightforward:

$$\theta_2'' + \big(\theta_1'\theta_1^{-1}\big)'\theta_2 + \big(\theta_1'\theta_1^{-1}\big)\theta_2' = 2(\beta + 1)\theta_1'.$$

Again using the Toda and the first identity we finally obtain

$$\theta_2''\theta_2^{-1} + \theta_2\phi^{-1} - \phi\psi - (\phi'\phi^{-1})^2 = 0$$

and then

$$\theta_2''\theta_2^{-1} + 2x - 2(\phi\psi + (\phi'\phi^{-1})^2) = \theta_2''\theta_2^{-1} - 2(x - \theta_2\phi^{-1}) = 0.$$

We will discuss one more step, namely the passage from $n = 2$ to $n = 3$ (then the recurrence will be clear). We want to show that

(1) $\theta_3'\theta_3^{-1} + \theta_2'\theta_2^{-1} = 2(\beta + 2)\theta_2\theta_3^{-1}$;
(2) $\theta_3''\theta_3^{-1} = 2\big(x - \theta_3\theta_2^{-1}\big)$.

From the second Toda and second identity we get

$$\theta_3 = 2x\theta_2 - \theta_2\theta_1^{-1}\theta_2 - \theta_2'\theta_2^{-1}\theta_2'.$$

It implies

$$\theta_3' = 2\theta_2 + 2x\theta_2' - \theta_2'\theta_1^{-1}\theta_2 + \theta_2\theta_1^{-1}\theta_1'\theta_1^{-1}\theta_2 - \theta_2\theta_1^{-1}\theta_2'$$
$$- 2\left(x - \theta_2\theta_1^{-1}\right)\theta_2' + \left(\theta_2'\theta_2^{-1}\right)^2\theta_2' - \theta_2'\theta_2^{-1}\left(2x - 2\theta_2\theta_1^{-1}\right)\theta_2.$$

We simplify and obtain from this

$$\theta_3' = 2\theta_2 + \theta_2\theta_1^{-1}\left(\theta_2' + \theta_1'\theta_1^{-1}\theta_2\right) + \theta_2'\theta_1^{-1}\theta_2 + \left(\theta_2'\theta_2^{-1}\right)^2\theta_2' - 2\theta_2'\theta_2^{-1}x\theta_2.$$

By the identity for θ_2' we have

$$\theta_3' = 2\theta_2 + \theta_2\theta_1^{-1}\cdot 2(1+\beta)\theta_1 + \theta_2'\theta_1^{-1}\theta_2 + \theta_2'\theta_2^{-1}\left(-\theta_3 - \theta_2\theta_1^{-1}\theta_2\right),$$

which assure the first identity for $n = 3$.

Now we prove the second.

Set $a = 2(\beta + 2)$. We have

$$\theta_3' = a\theta_2 - \left(\theta_2'\theta_2^{-1}\right)\theta_3.$$

Take the second derivation:

$$\theta_3'' = a\theta_2' - \left(\theta_2'\theta_2^{-1}\right)'\theta_3 - \theta_2'\theta_2^{-1}\theta_3'.$$

By using the formula for θ_3' we have

$$\theta_3'' = a\theta_2' - \left(\theta_2'\theta_2^{-1}\right)'\theta_3 - \theta_2'\theta_2^{-1}\left(a\theta_2 - \theta_2'\theta_2^{-1}\theta_3\right).$$

The terms with a are canceled and we have

$$\theta_3'' = -\left(\theta_2'\theta_2^{-1}\right)'\theta_3 + \left(\theta_2'\theta_2^{-1}\right)^2\theta_3.$$

Note that

$$-\left(\theta_2'\theta_2^{-1}\right)' + \left(\theta_2'\theta_2^{-1}\right)^2 = \theta_2''\theta_2^{-1} - 2\left(\theta_2'\theta_2^{-1}\right)'.$$

We already know that the first summand on the right-hand side equals $2\left(x - \theta_2\theta_1^{-1}\right)$ and by our Toda system

$$\left(\theta_2'\theta_2^{-1}\right)' = \theta_3\theta_2^{-1} - \theta_2\theta_1^{-1}$$

we obtain the second identity for θ_3.

The nth step of the recurrence goes as follows: from the nth Toda and recurrence conjecture we have

$$\theta_{n+1} = 2x\theta_n - \theta_n\theta_{n-1}^{-1}\theta_n - \theta_n'\theta_n^{-1}\theta_n'.$$

It implies

$$\theta_{n+1}' = 2\theta_n + 2x\theta_n' - \theta_n'\theta_{n-1}^{-1}\theta_n + \theta_n\theta_{n-1}^{-1}\theta_{n-1}'\theta_{n-1}^{-1}\theta_n - \theta_n\theta_{n-1}^{-1}\theta_n'$$
$$- 2\left(x - \theta_n\theta_{n-1}^{-1}\right)\theta_n' + \left(\theta_n'\theta_n^{-1}\right)^2\theta_n' - \theta_n'\theta_n^{-1}\left(2x - 2\theta_n\theta_{n-1}^{-1}\right)\theta_n.$$

Then, after some simplifications we get

$$\theta_{n+1}' = 2\theta_n + \theta_n\theta_{n-1}^{-1}\left(\theta_n' + \theta_{n-1}'\theta_{n-1}^{-1}\theta_n\right) + \theta_n'\theta_{n-1}^{-1}\theta_n + \left(\theta_n'\theta_n^{-1}\right)^2\theta_n' - 2\theta_n'\theta_n^{-1}x\theta_n.$$

By the recurrent formula for θ_n', we have

$$\theta_n' + \theta_{n-1}'\theta_{n-1}^{-1}\theta_n = 2(\beta + 1 - n)\theta_{n-1}$$

and

$$\begin{aligned}\theta'_{n+1} &= 2\theta_n + 2(\beta+n-1)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n + \left(\theta'_n\theta_n^{-1}\right)^2\theta'_n - 2\theta'_n\theta_n^{-1}x\theta_n \\ &= 2(\beta+n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n + \theta'_n\theta_n^{-1}\left(\theta'_n\theta_{n-1}^{-1}\theta'_n - 2x\theta_n\right) \\ &= 2(\beta+n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n + \theta'_n\theta_n^{-1}\left(-\theta_{n+1} - \theta_n\theta_{n-1}^{-1}\theta_n\right) \\ &= 2(\beta+n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n - \theta'_n\theta_n^{-1}\theta_{n+1} - \theta'_n\theta_{n-1}^{-1}\theta_n.\end{aligned}$$

which assure the first identity for $n+1$.

We leave the proof of the second identity for any n as an easy (though a bit lengthy) exercise similar to the case $n = 3$ above.

Identities (3) and (4) can be proved in a similar way. □

Lemma 3.4. *For* $n = 1$ *the left logarithmic derivative* $\phi'\phi^{-1} =: u_1$ *satisfies to* $nc - P_{II}(x, \beta)$.

Proof. From the previous lemma we have from the first Toda equation

$$(\phi'\phi^{-1})' = \theta_2\phi^{-1} - \phi\psi = \phi''\phi^{-1} - (\phi'\phi^{-1})^2 = 2(x - \phi\psi) - u_1^2$$

and hence

$$\theta_2\phi^{-1} = 2x - \phi\psi - u_1^2.$$

On the other hand, taking the derivative of the first Toda, we get

$$u_1'' = (\theta_2\phi^{-1} - \phi\psi)' = \theta_2'\phi^{-1} - \theta_2\phi^{-1}u_1 - (\phi'\psi + \phi\psi').$$

We replace $\theta_2'\phi^{-1}$ by

$$2(\beta+1) - u_1\theta_2\phi^{-1} = 2(\beta+1) - u_1\left(2x - \phi\psi - u_1^2\right).$$

Finally we obtain

$$u_1'' = 2u_1^3 - 2u_1x - 2xu_1 + 2(\beta+1) + u_1\phi\psi + \phi\psi u_1 - (\phi'\psi + \phi\psi'),$$

but

$$u_1\phi\psi + \phi\psi u_1 - (\phi'\psi + \phi\psi') = \phi\psi\phi'\phi^{-1} - \phi\psi' = 2\beta$$

which gives the desired result. □

Our proof of theorem 3.2 in the general case almost *verbatim* repeats the proof of lemma 3.4

Proof of theorem 3.2. Let $u_n := \theta'_n\theta_n^{-1}$. Now the same arguments, from the lemma 3.4, show that

(a) $\theta_{n+1}\theta_n^{-1} = 2x - \theta_n\theta_{n-1}^{-1} - u_n^2$;
(b) $\theta'_{n+1}\theta_n^{-1} = 2(\beta+n) - \theta'_n\theta_n^{-1}\theta_{n+1}\theta_n^{-1}$;
(c) $u_n'' = 2u_n^3 - 2xu_n - 2u_nx + 2(\beta+n) + \theta_n\theta_{n-1}^{-1}\left(\theta'_n\theta_n^{-1} + \theta'_{n-1}\theta_{n-1}^{-1}\right)$. This implies that $u_n'' = 2u_n^3 - 2xu_n - 2u_nx + 4\left(\beta + n - \frac{1}{2}\right)$. □

Remark. Using identities (3) and (4) from lemma 3.3 we can prove the second statement of theorem 3.2.

4. Discussion and perspectives

We have developed an approach to integrability of a fully noncommutative analog of the Painlevé equation. We construct solutions of this equation related to the 'fully noncommutative' Toda chain, generalizing the results of [EGR1, GR2]. This solutions admit an explicit description in terms of Hankel quasideterminants.

We consider here only the noncommutative generalization of Painlevé II but it is not difficult to write down some noncommutative analogs of other Painlevé transcendants. It is interesting to study their solutions, noncommutative τ-functions, etc. We hope that our equation (like its 'commutative' prototype) is a part of a whole noncommutative Painlevé hierarchy which relates (via a noncommutative Miura transform) to the noncommutative m-KdV and m-KP hierarchies (see i.e. [GNS, GN, EGR2, EGR1]). Another interesting problem is to study a noncommutative version of isomonodromic transformation problem for our Painlevé equation. The natural approach to this problem is a noncommutative generalization of generating functions, constructed in [JKM]. The noncommutative 'non-autonomous' Hamiltonian should be studied more extensively. It would be interesting to find noncommutative analogs of Okamoto differential equations [OK] and to generalize the description of Darboux–Bäcklund transformations for their solutions.

We address these and other open questions in the forthcoming papers.

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