Optical solitons in the few-cycle regime: Recent theoretical results

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OPTICAL SOLITONS IN THE FEW-CYCLE REGIME: RECENT THEORETICAL RESULTS

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Abstract. A brief overview of recent theoretical studies of several models used for the adequate description of both temporal and spatiotemporal dynamics of few-cycle optical pulses in both cubic and quadratic nonlinear media beyond the framework of slowly varying envelope approximation is given.

Key words: Ultrashort optical pulses, Few-cycle optical solitons, Circularly-polarized few-cycle pulses, Wave-polarization effects

1. INTRODUCTION

Ultrashort optical pulses comprising merely a few electric field oscillation cycles, are a matter of intensive current research activity since their first experimental realization in 1999 by several research groups [1]-[4]; see also a comprehensive early review on ultraintense few-cycle laser fields [5]. Such ultrashort pulses find diverse applications in the area of light matter interactions, high-order harmonic generation, extreme [6] and single-cycle [7] nonlinear optics, and attosecond physics [8]-[9]. The availability of ultrashort and ultraintense laser pulses generated by the powerful technique of chirped pulse amplification along with the development of high-fluence laser materials has opened up the field of optics in the relativistic regime [10]. Thus the ultrahigh electromagnetic field intensities I produced by these techniques ($I > 10^{18}$ W/cm^2), lead to relativistic effects generated by the motion of electrons in such laser fields, see, e.g., Ref. 10.

From the fundamental point of view, other interesting physical phenomena involving ultrashort optical pulses (with very broad spectra) are studied in detail at present. We mention here the supercontinuum generation (the spectral width exceeds two octaves) in microstructured photonic crystal fibers, which is seeded by femtosecond pulses in the anomalous group velocity dispersion regime of such fibers where unique physical processes such as soliton fission, stimulated Raman scattering, and dispersive wave generation were studied in detail, see, e.g., Refs. [11]-[13] and two comprehensive recent reviews [14]-[15].

It is worthy to mention here a recent experimental work demonstrating the synthesis of a single cycle of light by using compact erbium-doped fiber technology [16]; the obtained pulse duration of only 4.3 fs was close to the shortest possible value for a data bit of information transmitted in the near-infrared spectrum of light, at the wavelength of 1300 nm, see Ref. [16].

The traditional *slowly varying envelope approximation* (SVEA) is no longer valid for ultrashort optical pulses with duration of only a few femtoseconds. Although several generalizations of the SVEA have been proposed and have proven their efficiency [these generalizations are refered to as higher-order nonlinear Schrödinger equation (NLS) models, see e.g. Refs. [17]-[22]], a completely different approach to the study of few-cycle pulses (FCPs), which completely abandons the SVEA was put forward in a series of published works during the past two decades.

First, we mention that first-order nonlinear evolution equations can be obtained under the so-called *unidirectional approximation*. Non-SVEA models were proposed within the framework of the unidirectional approximation, see, e.g., Refs. [13] and [23]. Second, to the best of our knowledge, the necessity of using the non-SVEA approach for the adequate description of FCPs was put forward in the early seminal work by Akhmediev, Mel'nikov and Nazarkin published in 1989 [24]. In a subsequent paper, Belenov and Nazarkin [25] obtained exact solutions of nonlinear optics equations outside the approximation of slowly varying amplitudes and phases for light pulses a few wavelengths long and with power densities of the order of 10^9-10^{18} W/cm^2, clearly stating that traditional SVEA methods "are becoming ineffective in describing wave processes at such small spatial and temporal scales and at such high fields". However, several additional works introducing non-SVEA models for FCPs in different physical setting were published during the past two decades [26]-[44].

It is worthy to mention here some recent works on FCPs dealing with few-cycle spatiotemporal optical solitons (alias "light bullets" [45]-[49]) created by femtosecond filaments [50], the study of ultrashort light bullets in quadratic nonlinear media [51], the ultrashort spatiotemporal optical pulse propagation in cubic (Kerr-like) media without the use of the SVEA [52]-[53], single-cycle gap solitons generated in resonant two-level dense media with a subwavelength structure [54], observation of few-cycle propagating surface plasmon polariton wavepackets [55], and the possibility of generating few-cycle dissipative optical solitons [56]-[57]. We also mention recent studies of ultrafast pulse propagation in a mode-locked laser cavity in the few femtosecond pulse regime and the derivation by Farnum and Nathan Kutz [58] of a master mode-locking equation for ultrashort pulses. As clearly stated in this relevant recent work, the standard NLS-based approach of ultrafast pulse propagation, though has been shown "to work quantitatively beyond its expected breakdown, into the tens of femtoseconds regime, and has been used extensively for modeling supercontinuum generation … when pushed to the extreme of a few femtosecond pulses, the NLS description becomes suspect…" [58]. Other relevant works deal with the experimental study of intrinsic chirp of single-cycle pulses [59], and the proposal of a method to generate extremely short unipolar half-cycle pulses based on resonant propagation of a few-cycle pulse through asymmetrical media with periodic subwavelength structure [60].

Since 2003 [34], by using the *reductive perturbation method* or multiscale expansion (for a tutorial review of this powerful method, which is widely used in soliton theory, see Ref. [61]), a systematic analysis of the Maxwell-Bloch-Heisenberg equations describing the propagation of ultrashort (few-cycle) pulses in nonlinear optical media, put forward universal equations such as sine-Gordon (sG), Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), modified Korteweg-de Vries-sine Gordon (mKdV-sG), or Kadomtsev-Petviashvili (KP) ones [42], [51]-[53]. We notice that our earlier studies in this area [30]-[32], [36]-[37] were then generalized to a physical system consisting of two atomic transitions, one below and one above the range of propagated wavelengths. As a result, a model of mKdV-sG-type equations was put forward [33], [35], [38]. We notice that in certain cases this nonlinear dynamical system is completely integrable by means of the inverse scattering transform (IST) method. It admits stable soliton solutions of "breather" type, which also give a good account of few cycle soliton propagation. The complete integrability allowed us to investigate the interaction of FCP solitons and it was found that no phase matching is required [41].

The propagation of few-cycle pulses in a quadratic nonlinear medium has also been described by either a Korteweg-de Vries (KdV) or a Kadomtsev-Petviashvili equation, in (1+1)- or (2+1)- dimensional models, respectively, see Refs. [62] and [51]. In Ref. [62] it was shown that a FCP launched in a quadratically nonlinear optical medium may result in a half-cycle soliton in the form of a single hump, with no oscillating tails. By using the powerful reductive perturbation method a KdV equation was derived from both a classical and a quantum mechanical simple model of radiation-matter interaction [62]. These single-humped pulses always have a zero carrier-envelope phase, in the sense that the field polarity is completely determined by the properties of the medium, and as a direct consequence of this fact, the mean value of the optical electric field is not zero in such situations.

Moreover, by using the multiscale analysis, a generic Kadomtsev-Petviashvili evolution equation governing the propagation of femtosecond spatiotemporal optical solitons in quadratic nonlinear media beyond the SVEA was put forward [51]. Direct numerical simulations showed the formation, from adequately chosen few-cycle input pulses, of both stable *line solitons* in the case of a quadratic medium with normal dispersion and of stable *lumps* for a quadratic medium with anomalous dispersion. Notice that the perturbed unstable line soliton typically decays into stable lumps in the case of a quadratic nonlinear medium with anomalous dispersion [51].

In a recent work, we also found circularly polarized (CP) few-optical-cycle solitons in cubic nonlinear media in the long-wave-approximation regime and beyond the SVEA [63]. We found by direct numerical simulations that the CP few-optical-cycle soliton becomes unstable when the angular frequency is less than 1.5 times the inverse of the pulse duration, see Ref. 63.

The present work briefly summarizes the above mentioned new results in the area of both temporal and spatiotemporal dynamics of FCPs beyond the SVEA. In the next section we briefly overview the most general SVEA model based on the mKdV-sG nonlinear evolution equation. In Sec. 3 we consider FCP propagation in cubic nonlinear optical media. The propagation of FCPs in quadratically nonlinear optical media is briefly discussed in Sec. 4, where we point out the relevance of both KdV and KP evolution equations for the adequate description of such quadratic FCPs. The recent studies of circularly polarized (vectorial) FCPs beyond the slowly varying envelope approximation framework and the corresponding wave polarization effects is briefly overviewed in Sec. 5. Finally, in Sec. 6 we summarize the obtained results and we indicate a few possible extensions of these studies to other relevant physical settings.

2. THE MOST GENERAL SVEA MODEL (THE MKDV-SG NONLINEAR EVOLUTION EQUATION)

The generic mKdV-sG equation is written as:

$$E_z + c_1 \sin\left(\int E\right) + c_2\left(E^3\right)_t + 2c_2 c_3 E_{ttt} = 0. \tag{1}$$

The mKdV-sG equation (1) can be derived from Maxwell-Bloch equations, and describes FCP soliton propagation in Kerr (cubic) optical media. It reduces to mKdV equation if $c_1 = 0$, and to sG equation if $c_2 = c_3 = 0$. The two latter equations are completely integrable by means of the IST, and Eq. (1) is also completely integrable if $c_3 = 2c_2$ [64]. It admits breather solutions, which describe FCP solitons (see Fig. 1 for a typical breather solution of the mKdV-sG equation (1)).

Notice that during the past several years other non-SVEA models have been proposed to describe FCP soliton propagation. Among them is the so-called short-pulse equation (SPE) [65]:

$$E_{zt} = E + \frac{1}{6}\left(E^3\right)_{tt}. \tag{2}$$

It is well known that the SPE (2) is completely integrable [66], and it accounts for FCP soliton propagation. Notice that the SPE can be derived from mKdV-sG [42]. In order to prove this assertion let us first perform a small amplitude approximation, so that the sine term in Eq. (1) reduces to $c_1 \int^t u$; then the mKdV-type dispersion is neglected: $c_3 = 0$. A linear change of variables allows to fix the values of the remaining coefficients to $c_1 = -1$, $c_2 = -1/6$, and derivative with respect to t yields exactly the SPE equation (2). If we use the same small amplitude approximation, but do not neglect the mKdV-type term, we obtain after rescaling ($c_3 = -\mu$ and $c_1 = c_2 = 1$) the alternative model equation

$$E_{zt} + E - \mu E_{tttt} + \left(E^3\right)_{tt} = 0. \tag{3}$$

Equation (3) was first proposed to model FCP soliton propagation in Ref. [67] and it has shown FCP pulse compression, see Refs. [68] and [39]. Hence we see that all non SVEA models which have been put forward in the literature, in order to model FCP propagation in nonlinear optical media are obtained as certain approximations of the generic mKdV-sG equation (1).

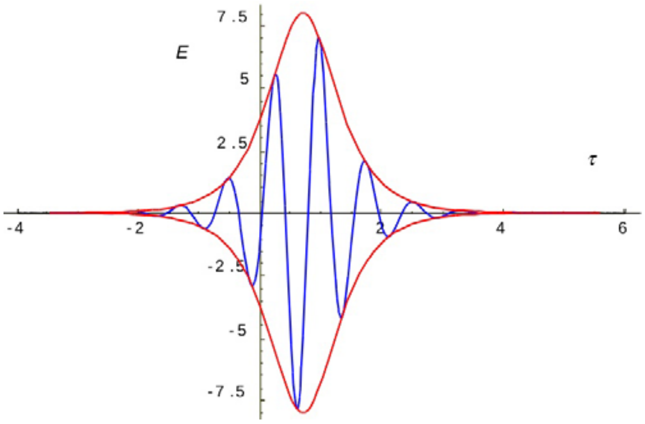

Fig. 1 - A typical FCP soliton described by the breather solution of the mKdV-sG equation (1). Blue (dark gray): the analytical profile, red (light gray): its exact envelope; after Ref. [44].

3. FCP SOLITONS IN CUBIC NONLINEAR MEDIA

Next we consider the defocusing-type mKdV equation. Notice that there are actually two different mKdV equations:

$$u_z + \sigma u^2 u_t + u_{ttt} = 0\,, \qquad (4)$$

where $\sigma = \pm 1$. For $\sigma = +1$ the mKdV equation (4) is of focusing type, while for $\sigma = -1$ it is of defocusing type. The mKdV-sG equation with a defocusing mKdV part supports FCP solitons, see Ref. [42] for details. This fact can be argued as follows: for high frequencies (closer to SVEA limit), the mKdV-sG equation can be approximately mapped to sG equation [42]. Then the exact sG breather solution allows us to construct an approximate soliton, which can be used as input in a numerical resolution of mKdV-sG equation. It evolves with little shape deformation, as is shown in Fig. 2 in the particular case of vanishing mKdV dispersion. For really defocusing mKdV dispersion, pulse compression may occur, as shown in Fig. 3. The input waveform used is this numerical simulation is the soliton of the nonlinear Schrödinger equation which corresponds to the SVEA limit of mKdV-sG equation; it is seen that SVEA limit is not valid, see Ref. [42].

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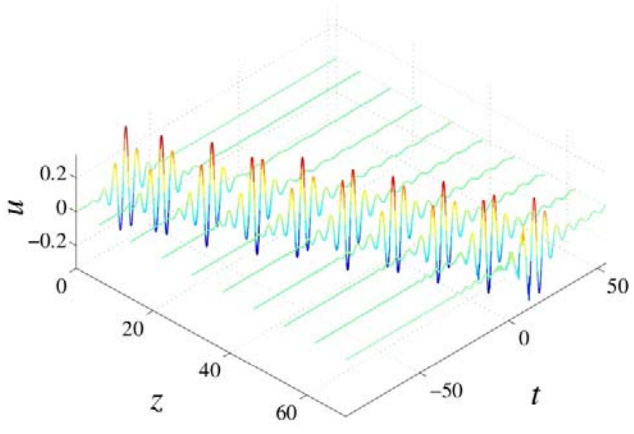

Fig. 2 – Typical evolution of a FCP soliton according to the mKdV-sG equation with defocusing mKdV nonlinearity and vanishing mKdV dispersion; after Ref. [42].

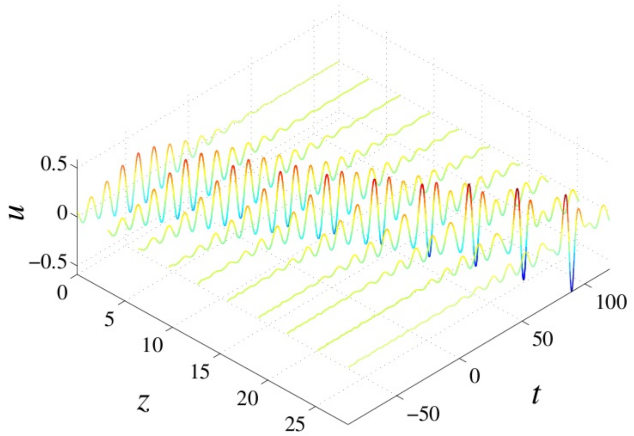

Fig. 3 – Typical pulse compression according to the mKdV-sG equation with defocusing mKdV part; after Ref. [42].

4. FCP SOLITONS IN QUADRATIC NONLINEAR OPTICAL MEDIA

In a previous work [62] it was proved that starting from either a classical model of elastically bound electrons or a quantum two-level model, in which a quadratic optical nonlinearity has been introduced, the reductive perturbation method allowed us to derive a KdV equation:

$$\partial_{\zeta} E = A\partial_{\tau}^{3} E + B\partial_{\tau}\left(E\right)^{2} , \qquad (5)$$

in which the dispersion coefficient A is

$$A = \frac{1}{6}\frac{d^3k}{d\omega^3}\bigg|_{\omega=0} = \frac{1}{2c}\frac{d^2n}{d\omega^2}\bigg|_{\omega=0}, \tag{6}$$

and the nonlinear coefficient B is given by

$$B = -\frac{2\pi}{nc}\chi^{(2)}\left(2\omega;\omega,\omega\right)\Big|_{\omega=0}. \tag{7}$$

It was found that a quadratic FCP soliton can be formed from an adequate FCP input, from both the resolution of the KdV equation (5) by the IST and numerical analysis, see Ref. [62]. The ultrashort soliton is exactly a half-cycle one, with no oscillating tail. In addition, it has a determined polarity. However, a large part of energy is dispersed, and soliton formation strongly depends on initial carrier-envelope phase, see Ref. [62].

Notice that in (2+1) dimensions, the generic KdV equation becomes the Kadomtsev-Petviashvili (KP) equation, either KP I or KP II. For the normal dispersion case, it is the so-called KP II equation, which admits stable *line solitons*. This corresponds to a nonlinear recovery of the initial wavefront, hence the spatial coherence of the wave is improved by the nonlinear effect (see Fig. 4 for a typical example of the formation of line solitons). For the anomalous dispersion case, the governing model is the so-called KP I equation, which admits stable localized *lump solutions* [69]-[70]. The exact, analytical profile of the lump soliton is shown in Fig. 5. Numerical simulations show that lump solitons form spontaneously from transverse irregularities, see Ref. [51].

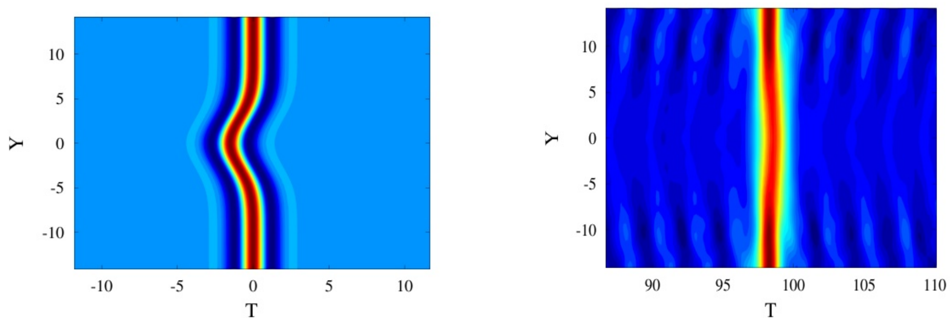

Fig. 4 – Typical FCP in quadratic nonlinear media with normal dispersion: Recovery of a perturbed wavefront according to the KP II equation and generation of a line soliton. Left: input, right: after propagation; after Ref. [51].

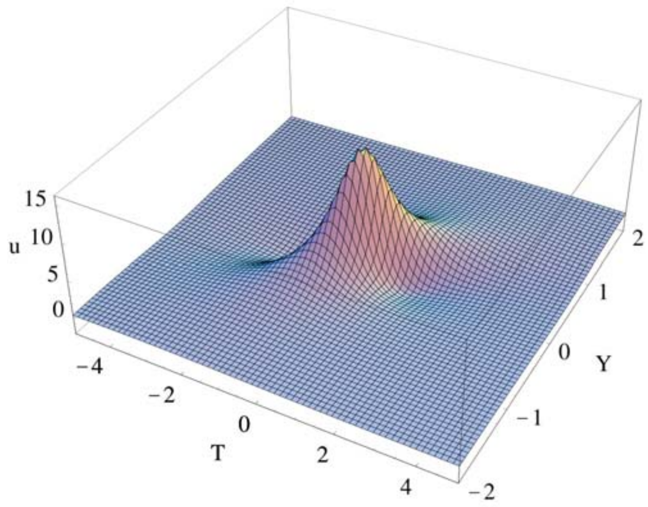

Fig. 5 – Typical two-dimensional FCP soliton (lump soliton) in quadratic nonlinear media with anomalous dispersion, as given by the analytical solution; after Ref. [51].

5. VECTORIAL FCP SOLITONS: POLARIZATION EFFECTS

Recently [63] we considered the propagation of CP FCPs, i.e., of vectorial FCPs in Kerr media beyond the slowly varying envelope approximation. Assuming that the frequency of the transition is far above the characteristic wave frequency (i.e., the so-called *long-wave-approximation regime*), we showed that propagation of FCPs, taking into account the wave polarization, is described by the nonintegrable complex modified Korteweg–de Vries (cmKdV) equation, see Ref. [63] for details. By direct numerical simulations, we got robust localized solutions to the cmKdV equation, which describe CP few-cycle-optical solitons. These robust localized solutions strongly differ from the breather solitons of the modified Korteweg–de Vries equation, which represents linearly polarized (LP) FCP solitons. The CP FCP soliton becomes unstable when its angular frequency is less than 1.5 times the inverse of the pulse length [63]. Moreover, we found by direct numerical simultations that the unstable subcycle pulses decay into LP half-cycle pulses, the polarization direction of which slowly rotates around the propagation axis, see Ref. [63].

In the following we show a typical example of the robust propagation of a CP soliton. As we said above we work in the long-wave-approximation regime. In this case, the typical frequency ω_w of the wave must be far away from the resonance frequency Ω of the two-level atoms, because the transparency of the medium is required for soliton propagation. Thus we consider in what follows the situation $\omega_w \ll \Omega$., i.e., the typical duration of the wave, let say, $t_w = 1/\omega_w$, is very large with respect to the characteristic time $t_r = 1/\Omega$ associated to the transition.

By using the powerful multiscale analysis [63], we got the following pair of coupled nonlinear evolution equations in their normalized form, for the x- and y-polarized electric field components U and V, respectively:

$$U_Z - U_{TTT} - [(U^2 + V^2)U]_T = 0 \tag{8}$$

$$V_Z - V_{TTT} - [(U^2 + V^2)V]_T = 0 \tag{9}$$

Here Z and T are the normalized propagation distance and the retarded time, respectively. Notice that the retarded normalized time variable T describes the pulse shape propagating at speed V=c/n, where n is the refractive index. Next, setting f=U+iV, Eqs. (8) and (9) reduce to the the following complex cmKdV equation (the so-called cmKdV I equation), which is not a completely integrable one:

$$f_Z - f_{TTT} - (|f|^2 f)_T = 0 \tag{10}$$

An approximate solution to the above cmKdV I equation is given by:

$$f(T,Z) = \sqrt{6}\, b \,\mathrm{sech}\left[b\left(T - 3\omega^2 Z\right)\right] e^{i\omega\left[T-\left(\omega^2-3b^2\right)Z\right]} \tag{11}$$

The above formula is valid for long pulses, i.e., for b « ω. In Fig. 6 we show the evolution of a FCP of the above form, with b = 1 and ω = 2. The robustness of the FCP is obvious, and its width and maximum amplitude are conserved, see Fig. 6. Also, we have investigated in Ref. [63] the transition to a half-cycle soliton for sufficiently small values of the ratio ω/b. Thus we have found that the value $\omega / b \simeq 1.5$ is the lower limit for the stability of the CP FCP soliton. For values of the ratio ω/b less than 1.5, the FCP becomes unstable and decays into a LP single-humped (half-cycle) pulse in the form of a fundamental soliton of the real mKdV equation; see Ref. [63].

We also considered in other recent work [69] the propagation of FCPs beyond the SVEA, in media in which the dynamics of constituent atoms is described by a two-level Hamiltonian, by taking into account the wave polarization. We considered the *short-wave approximation*, assuming that the resonance frequency of the two-level atoms is well below the inverse of the characteristic duration of the optical pulse. By using the reductive perturbation method (multiscale analysis) we derived from the Maxwell-Bloch-Heisenberg equations the governing evolution equations for the two polarization components of the electric field in the first order of the perturbation approach. We showed that propagation of CP few-optical-cycle solitons is described by a system of coupled nonlinear equations, which reduces in the scalar case to the standard sine-Gordon equation describing the dynamics of LP FCPs in the short-wave-approximation regime. We also calculated by direct numerical simulations the lifetime of CP FCPs and we investigated the transition to two orthogonally polarized single-humped pulses as a generic route of their instability; see Ref. [69] for more details of this study.

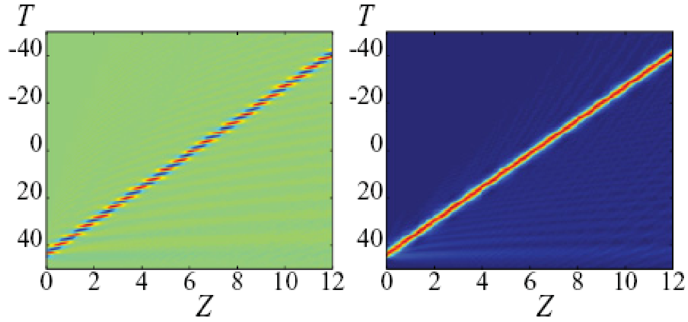

Fig. 6 – Propagation of a CP FCP, which shows its remarkable robustness. Left panel: x-polarized component. Right panel: Norm of the electric field |f |. The input is given by Eq. (9) with b = 1 and ω = 2; after Ref. [63].

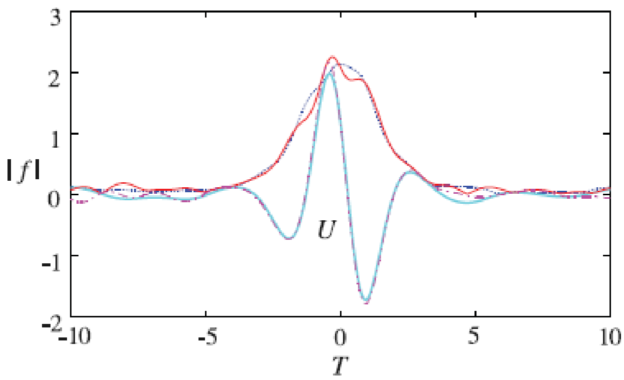

Fig. 7 – Initial (Z = 100) and final (Z = 10 000) profiles of the FCP plotted in Fig. 6 for the input given by Eq. (11). Blue (dotted line): initial |f |; light blue (thick gray line): initial U; red (thin solid line): final |f |; pink (dashed-dotted line): final U; after Ref. [63].

6. CONCLUSIONS

We developed in the past few years an adequate theory of ultrashort (femtosecond) soliton propagation beyond the commonly used SVEA for both Kerr and quadraticaly nonlinear optical media. We have summarized in this overview some recent results of this self-consistent theory, emphasizing the generality and interest of the generic mKdV-sG model in the case of cubic (Kerr) nonlinear media. Also, some aspects of FCP soliton propagation in quadratic nonlinear media have been briefly discussed. In both cases, no phase matching is required, which makes a strong contrast with the longer (picosecond) pulses described by the common SVEA. In addition we have briefly overviewed our recent work on these general lines concerning the vectorial (circularly polarized) FCPs in both the long-wave and the short-wave limits. Therefore we put forward, in these recent studies, unique polarization effects of ultrashort solitons with duration of only few optical cycles.

A natural extension suggested by these studies is to consider the case of two optical transitions, one below and one above the range of propagated wavelengths. Another interesting issue might be the generalization of these works to one or even to two spatial transverse dimensions, in addition to time and spatial longitudinal coordinates, that is, to investigate, beyond the slowly varying envelope approximation, vectorial few-optical-cycle spatiotemporal solitons (ultrashort circularly polarized light bullets) and the associated wave-polarization effects.

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