Definition of optimal accelerated test plan

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***Abstract*— The objective of this paper is to define an optimal accelerated test plan considering an economic approach. The objective function is defined by two terms: the cost linked to testing activities and the cost associated to operation of the product. The optimal test plans are defined in considering a prior knowledge on reliability parameters (choice of reliability function, scale and shape parameters …) and acceleration model (choice of model, model parameters …) to characterize the accelerated life model. This information is used in Bayesian inference (to optimize the testing plan). The prior knowledge contains also the uncertainty on real reliability of new product. So, the developed methodology proposes to define an optimal accelerated testing plan in considering an objective function based on economic approach, Bayesian inference for optimizing the test plan and taking into account the uncertainty on parameters to obtain a robust optimal test plan.**

Keywords- accelerated test, reliability, optimisation, Bayesian estimation, testing cost, operation cost, robustness.

I. INTRODUCTION

Accelerated Life Test (ALT) is a test method which subjects test units to higher than use stress levels in order to compress the time to failure of the units. Conducting a Quantitative Accelerated Life Test (QALT) requires the determination or development of an appropriate life-stress relationship model as illustrated in Figure 1. below.

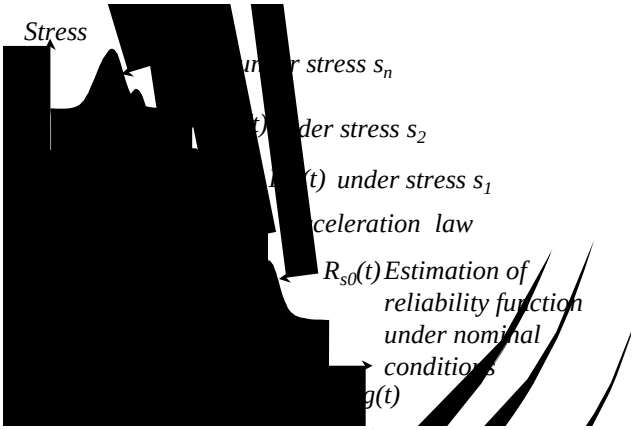


Figure 1. Principle of ALT test

Moreover, a test plan needs to be developed to obtain appropriate and sufficient information in order to accurately estimate reliability performance at operating conditions, significantly reduce test times and costs and achieve other objectives. One of the first decisions to be made when designing a reliability verification test is to determine how many units to test. If many units are tested, the duration of the test will be short. With this approach, prototype costs will be high, and development time costs will be low. If few units are tested, the duration of the test will be longer: prototype costs will be low, but development time costs will be high.

In this paper, a methodology is proposed to define an optimal accelerated test plan in order to maximize the accuracy on the estimate of the statistical distribution of the life spans under the nominal conditions. The main development concerns the definition of optimal accelerated test plan considering an economic approach. The objective function is defined by two terms: the cost linked to testing activities and the cost associated to operation of the product. The optimal test plans are defined in considering a prior knowledge on reliability parameters (choice of reliability function, scale and shape parameters …) and acceleration model (choice of model, model parameters …) to characterize the accelerated life model. This information is used in Bayesian inference (to optimize the testing plan). The prior knowledge contains also the uncertainty on real reliability of new product. So, the developed methodology proposes to define an optimal accelerated testing plan in considering an objective function based on economic approach, Bayesian inference for optimizing the test plan and taking into account the uncertainty on parameters to obtain a robust optimal test plan. The methodology is illustrated by a numerical example on accelerated test on ball bearing.

II. ESTIMATION IN PARAMETRIC ALT MODEL(SVA MODEL)

Assuming that the lifetime $T_{X(\bullet)}$ for no matter which stress $X(\bullet)$ is a non-negative random variable described by the reliability function:

$$R_{X(\bullet)}(t) = P\{T_{X(\bullet)} > t\} t \geq 0 \qquad (1.)$$

than, for a stress set ε, the stress $X_1(\bullet)$ is greater than a stress $X_0(\bullet)$ if $R_{X_0}(\bullet)(t) > R_{X_1}(\bullet)(t)$ for every $t > 0$ Having $\varepsilon_0 \subset \varepsilon$ a set of constant stress over time and $X_0 \in \varepsilon_0$, the inverse function of $R_{X_0}(t)$ can be defined as:

$$R_{X_0}^{-1}(p) = \inf\left\{t \ : \ R_{X_0}(t) \geq p\right\} \qquad (2.)$$

The convolution product between $R_{X_0}^{-1}$ and $R_X(\bullet)$ is denoted as $f_{X(\bullet)}(t) = R_{X_0}^{-1}\left(R_{X(\bullet)}(t)\right)$, with $f_{X(\bullet)}(0) = 0$. The standard ALT model is defined over ε if there is a function as for every $X(\bullet) \in \varepsilon$:

$$\frac{d}{dt} f_{X(\bullet)}(t) = r[X(t)] \qquad (3.)$$

The function *r* defines a degradation rate. The ALT model states that the rate of resource used at the moment *t* depends only on the value of the applied stress at the moment *t*. The equation 3 involves:

$$R_{X(\bullet)}(t) = R_{X_0}\left(\int_0^t r[X(\tau)]d\tau\right) \quad (4.)$$

A special case is $X(\tau) \equiv X = \text{const}$:

$$R_X(t) = R_{X_0}(r(X)t) \quad (5.)$$

So, the stress affects only the scale parameter. Note that $r(X_0)$=1. If the function *r* is completely unknown, the reliability function R_{x_0} cannot be estimated. Therefore, the function *r* is chosen in particular classes of function. In survival analysis, the log-linear models defining the ALT models are frequently used as regression models[1].

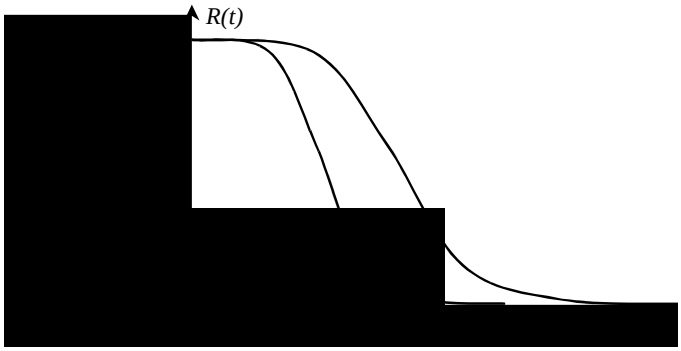


Figure 2. Definition of time transfer regression model r(x)

The log-linear models specify the effect of the covariates as, respectively, a multiplicative factor for the hazard rate, a scale change, or a location shift for the reliability function. As consequence, generalizing, and the model becomes:

$$R_{X(\bullet)}(t) = R_{X_0}\left(\int_0^t e^{\beta^T Z(\tau)} d\tau\right) \qquad (6.)$$

and, for the particular cases of constant stress, the equation 5 becomes (see figure 2):

$$R_X(t) = R_{X_0}\left(e^{\beta^T Z} t\right) \qquad (7.)$$

with $\beta = (\beta_0, ..., \beta_m)^T$ a parameters vector, $Z = (\varphi_0(X), .., \varphi_m(X))^T$ with the functions φ_i specified and with the first component Z_0 equal to 1. Several models, as Arrhenius, inverse power, generalized Eyring and so on, can be obtained as a particular case of this general form.

It is assumed that the survival function *R(t)* belongs to a class of functions depending only on the parameters of scale η and shape ν [7] :

$$R(t) = R\left(\left(\frac{t}{\eta}\right)^{\upsilon}\right), (\eta, \upsilon > 0) \qquad (8.)$$

Several models as Exponential, Weibull, lognormal, loglogistic and so on are just particular cases of the above form as detailed in [4].

The notations $R(t) = R\left(e^t\right)$, $t \in \Re$, $\sigma = \frac{1}{\upsilon}$, $\upsilon = \left(\upsilon_0, ..., \upsilon_m\right)$, $\gamma_0 = \ln\eta - \beta_0$ et $\gamma_i = -\beta_i$, $i = (1, ..., m)$ allow us to rewrite the equation 7 as:

$$R_X(t) = R\left(\frac{\ln t - \gamma^T Z}{\sigma}\right) \qquad (9.)$$

Imposing T_{ij} lifetime observed or censored for the *j*-th unit of *i*-th group, we have:

$T_{ij} = \ln\left(\mathbf{T}_{ij} \wedge t_i\right)$ with t_i the censoring time and $\wedge$ denoting the minimum between the terms

$$\delta_{ij} = I\left\{\mathbf{T}_{ij} \leq t_i\right\} = \begin{cases} 1 \text{ if observed time} \\ 0 \text{ if censoring time} \end{cases}$$

$$f(u) = -S'(u)$$

$$\lambda(u) = \frac{f(u)}{S(u)}$$

The likelihood function can be written as:

$$L(T \mid \gamma, \sigma) = \prod_{i=1}^{k}\prod_{j=1}^{n_i}\left\{\left[\frac{1}{\sigma}\lambda\left(\frac{T_{ij} - \gamma^T Z^{(i)}}{\sigma}\right)\right]^{\delta_{ij}} R\left(\frac{T_{ij} - \gamma^T Z^{(i)}}{\sigma}\right)\right\} \quad (10.)$$

A. Point Estimate

The likelihood principle: *all information on the parameters taken from a number of observations is finally contained in the likelihood* characterizes the likelihood as maximized for a set of parameters identified. For this, we consider that the failures

are independent, meaning that the failure of a system that does not affect another system [1].

If $L(\mathbf{T} \mid \theta)$ is differentiable and if the maximum likelihood $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_r)$ exists, then it satisfies the following equation:

$$\left. \frac{\partial L(t_1 ..t_n \mid \theta_1 ..\theta_r)}{\partial \theta_j} \right|_{\theta_j = \hat{\theta}_j} = 0 \qquad (11.)$$

The computation of $R(t \mid \theta = \hat{\theta})$ allows to approximate the reliability function associated with the product generating times of failure for the point estimate $\theta = \hat{\theta}$. An estimation of confidence intervals is required to complete the approach.

B. Confidence Intervals

The Cramer-Rao bound, or the lower limit of Cramer-Rao expresses a lower limit on the disagreement of a parameter estimators. In its simplest form, the bound states that the variance of any unbiased estimator is at least as high as the inverse of the Fisher information. An unbiased estimator, which achieves this lower bound, is said to be efficient. Such a solution achieves the lowest possible mean squared error among all unbiased methods, and is therefore the minimum variance unbiased (MVU) estimator.

Let $X(Y)$ be an estimator of any vector function of parameters, $X(Y) = (X_1(Y), \ldots X_n(Y))^T$, and denote its expectation vector $\mathbf{E}[X(Y)]$ by $\psi(\theta)$. If $X(Y)$ is an unbiased estimator of θ (i.e., $\psi(\theta) = \theta$), then the Cramer-Rao bound states:

$$\mathbf{cov}_\theta(X(Y)) \geq I(\theta)^{-1} \qquad (12.)$$

with $I(\theta) = E\left[\left(\frac{\partial \log L(\mathbf{T}|\theta)}{\partial \theta} \right)^2 |_{\theta=\hat{\theta}} \right]$ the Fisher information.

The confidence limits for the case of $\theta > 0$ are obtained by processing the $\log(\theta)$ as a normally distributed variable:

$$\hat{\theta} \times e^{-\frac{w_{\frac{1-\alpha}{2}} \sqrt{Var(\hat{\theta})}}{\hat{\theta}}} < \theta < \hat{\theta} \times e^{\frac{w_{\frac{1-\alpha}{2}} \sqrt{Var(\hat{\theta})}}{\hat{\theta}}} \qquad (13.)$$

Moreover, for most of the cases, typically values for the parameters can be found. The values may be considered either as results of expertise (using references as FIDES [5]), or as values associated to older/similar products or as results of single/several expert(s)' opinion. Whatever method might be used to obtain the prior information, an improvement of the final estimation of model's parameters can be attended by a diminution of confidence intervals over the estimations as result of increasing the Fisher information.

III. BAYESIAN INFERENCE

The classic statistics are based on the notion of objective probability. The real value is approached by a frequency associated with results obtained from a sample. Larger the size of a sample, closer the frequency calculated is to the true value of probability. This estimate is closely related to sampling.

On the contrary, the Bayesian approach is based on the concept of subjective probability depending on the degree of belief in the occurrence of an event. This is not a point value, which is estimated, but the probability distribution of the random variable *(*probability of non-functioning), the degree of belief that each probability value can be true.

It is tempting to see the probability density as an estimator whose Maximum Likelihood method (ML) would be the mode; such a position was, for example, supported by Laplace who believed that the absence of prior information justified the choice of the uniform distribution. Similarly, Fisher, introducing analysis trustee, wanted to implement the principle of likelihood without going through a Bayesian approach. The choice of distribution as it is then totally objective.

The continuous form of Bayes theorem for the random variable θ over the Ω domain, having t_i, $i = \overline{1..n}$ as test results, is:

$$\pi_{apo}(\theta / t_i) = \frac{L(t_1, .., t_n / \theta) \pi(\theta)}{\int_\Omega L(t_1, .., t_n / \theta) \pi(\theta) d\theta} \qquad (14.)$$

with $\pi(\theta)$ the mathematical form, which formalizes the prior information. The prior information over the normal conditions of use will be assimilated within the results of a number of tests already done hypothetically and added to the values of real tests.

By the assumption of independence between the variables, the joint distribution can be defined as:

$$\pi(\gamma, \sigma) = \pi_\sigma(\sigma) \left[\prod_{l=0}^{m} \pi_{\gamma_l}(\gamma_l) \right]$$

Integrating the likelihood function (equation 10) into the general form of Bayes' theorem (equation 14), the posterior distribution becomes:

$$\pi_{apo}(\gamma, \sigma | T) = \frac{\prod_{i=1}^{k} \prod_{j=1}^{n_i} \left[\frac{1}{\sigma} \lambda \left(\frac{T_{ij} - \gamma^T Z^{(i)}}{\sigma} \right) \right]^{\delta_{ij}}}{K_I} \frac{R\left(\frac{T_{ij} - \gamma^T Z^{(i)}}{\sigma} \right) \pi(\gamma, \sigma)}{Q_I} \qquad (15.)$$

with a normalization constant:

$$Q_I = \int_0^{\infty}\left\{\int_0^{\infty}..\int_0^{\infty}\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}\left[\frac{1}{\sigma}\lambda\left(\frac{T_{ij}-\gamma^T Z^{(i)}}{\sigma}\right)\right]^{\delta_{ij}} \times R\left(\frac{T_{ij}-\gamma^T Z^{(i)}}{\sigma}\right)\right]\pi(\gamma,\sigma)d\gamma_1\cdots d\gamma_m\right\}d\sigma$$

Conceptually, while ML adopts a classical approach, i.e., only experimental measurements are supplied to the estimator, Maximum a Posteriori (MAP) estimation is a Bayesian approach, i.e., prior available statistical information on the unknown parameters is also exploited for their estimation. The Bayes estimation can be of relevant interest, since it can significantly improve the precision of parameter estimates with respect to Fisher estimation. This may also allow the adoption of more complex models than those determinable by a Fisherian approach.

Still following the aspect of reversing in statistics, we shall consider the newly obtained $\pi_{apo}(\gamma,\sigma)$ function as a probability density and we shall treat it as. As consequence, the ML theory can be applied. So a search of values that maximizes the $\pi_{apo}(\gamma,\sigma)$ and the variances associated to these estimators will be searched.

The differentiating after γ_i and σ of the function $\ln\left[\pi_{apo}(\gamma,\sigma)\right]$ gives the terms:

$$\begin{aligned}U_l(\gamma,\sigma) &= \frac{\partial\left(\ln\left[\pi_{apo}(\gamma,\sigma|T)\right]\right)}{\partial\gamma_l} \\ &= \frac{\partial\ln\left[\frac{L(T|\gamma,\sigma)\pi(\gamma,\sigma)}{K_I}\right]}{\partial\gamma_l} \\ &= \frac{\partial\ln\left[L(T|\gamma,\sigma)\right]}{\partial\gamma_l}+\frac{\partial\ln\left[\pi(\gamma,\sigma)\right]}{\partial\gamma_l}-\frac{\partial\ln\left[K_I\right]}{\partial\gamma_l} \\ &= \frac{\partial\ln\left[L(T|\gamma,\sigma)\right]}{\partial\gamma_l}+\frac{\mathbf{d}\ln\left[\pi_{\gamma_l}(\gamma,\sigma)\right]}{\mathbf{d}\gamma_l} \\ &= \frac{1}{\sigma}\sum_{i=1}^{k} z_{il}\sum_{j=1}^{n_i} a_{ij}(\gamma,\sigma)+\frac{\mathbf{d}\ln\left[\pi_{\gamma_l}(\gamma,\sigma)\right]}{\mathbf{d}\gamma_l}\end{aligned}$$

with $(l=1,...,m)$ and:

$$\begin{aligned}U_{m+1}(\gamma,\sigma) &= \frac{\partial\left(\ln\left[\pi_{apo}(\gamma,\sigma|T)\right]\right)}{\partial\sigma} \\ &= \frac{\partial\ln\left[\frac{L(T|\gamma,\sigma)\pi(\gamma,\sigma)}{K_I}\right]}{\partial\sigma} \\ &= \frac{\partial\ln\left[L(T|\gamma,\sigma)\right]}{\partial\sigma}+\frac{\partial\ln\left[\pi(\gamma,\sigma)\right]}{\partial\sigma}-\frac{\partial\ln\left[K_I\right]}{\partial\sigma} \\ &= \frac{\partial\ln\left[L(T|\gamma,\sigma)\right]}{\partial\sigma}+\frac{\mathbf{d}\ln\left[\pi_{\sigma}(\gamma,\sigma)\right]}{\mathbf{d}\sigma} \\ &= \frac{1}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left\{v_{ij}(\gamma,\sigma)a_{ij}(\gamma,\sigma)-\delta_{ij}\right\}+\frac{\mathbf{d}\ln\left[\pi_{\sigma}(\gamma,\sigma)\right]}{\mathbf{d}\sigma} \quad (16.)\end{aligned}$$

with:

$$v_{ij}(\gamma,\sigma)=\frac{T_{ij}-\gamma^T Z^{(i)}}{\sigma}$$

$$a_{ij}(\gamma,\sigma)=\lambda\left(v_{ij}(\gamma,\sigma)\right)-\delta_{ij}\left(\ln\lambda\right)'\left(v_{ij}(\gamma,\sigma)\right)$$

$$\frac{\partial\ln\left[\pi_{\sigma}(\gamma,\sigma)\right]}{\partial\sigma}=\frac{\mathbf{d}\left(\ln\left[d\left(\frac{\sigma}{c}\right)^d\right]+\ln\left[f_0^{\{m+1\}}\left[\left(\frac{\sigma}{c}\right)^d\right]\right]\right)}{\mathbf{d}\sigma}$$

$$\frac{\partial\ln\left[\pi_{\gamma_l}(\gamma,\gamma_l)\right]}{\partial\gamma_l}=\frac{\mathbf{d}\left(\ln\left[b\left(\frac{\gamma_l}{a}\right)^b\right]+\ln\left[f_0^{\{l\}}\left[\left(\frac{\gamma_l}{a}\right)^b\right]\right]\right)}{\mathbf{d}\gamma_l}$$

The MAP estimators $\hat{\sigma},\hat{\gamma}$ can be obtained by solving the equation system:

$$U_p(\gamma,\sigma)=0 \ \ (p=1,...,m+1) \quad (17.)$$

Fisher information applies to the function that describes the information on the parameters, $\pi_{apo}(\gamma,\sigma|T)$, [6]:

$$\begin{aligned}I^{MAP}(\gamma,\sigma) &= E\left[\left(\frac{\partial\log\pi_{apo}(\gamma,\sigma|T|}{\partial\gamma\partial\sigma}\right)^2|_{\hat{\gamma},\,\hat{\sigma}}\right] \\ &= E\left[\left(\frac{\partial\log L(T|\gamma,\sigma|}{\partial\gamma\partial\sigma}\right)^2|_{\hat{\gamma},\,\hat{\sigma}}\right]+ \\ &\quad +E\left[\left(\frac{\partial\log\pi(\gamma,\sigma)}{\partial\gamma\partial\sigma}\right)^2|_{\hat{\gamma},\,\hat{\sigma}}\right] \\ &= I^L(\gamma,\sigma)+I_{\pi}^{AP}(\gamma,\sigma)\end{aligned} \quad (18.)$$

with:

$$\begin{aligned}I_{ls}^L(\gamma,\sigma) &= -\frac{\partial^2\ln L(T\,|\,\gamma,\sigma)}{\partial\gamma_l\partial\gamma_s} \\ &= \frac{1}{\sigma^2}\sum_{i=1}^{k} z_{il}z_{is}\sum_{j=1}^{n_i} c_{ij}(\gamma,\sigma)\end{aligned}$$

$$\begin{aligned}I_{l,m+1}^L(T\,|\,\gamma,\sigma) &= -\frac{\partial^2\ln L(\gamma,\sigma)}{\partial\gamma_l\partial\sigma} \\ &= \frac{1}{\sigma}U_l(\gamma,\sigma)+\frac{1}{\sigma^2}\sum_{i=1}^{k} z_{il}\sum_{j=1}^{n_i} v_{ij}(\gamma,\sigma)c_{ij}(\gamma,\sigma)\end{aligned}$$

$$I^{L}_{m+1,m+1}(\gamma,\sigma)=\frac{\partial^2 \ln L(T\mid\gamma,\sigma)}{\partial\sigma^2}$$

$$=\frac{2}{\sigma}U_{m+1}(\gamma,\sigma)+\frac{1}{\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(v_{ij}^2(\gamma,\sigma)c_{ij}(\gamma,\sigma)+\delta_{ij}\right) \quad (19.)$$

$$I^{AP}_{\pi}(\gamma,\sigma)=E\left[\left(\frac{\partial \log\pi(\gamma,\sigma)}{\partial\gamma\partial\sigma}\right)^2\mid_{\hat{\gamma},\hat{\sigma}}\right]$$

and: $c_{ij}(\gamma,\sigma)=\lambda'\left(v_{ij}(\gamma,\sigma)\right)-\delta_{ij}\ (\ln\lambda)''\left(v_{ij}(\gamma,\sigma)\right)$ and $l,s=0,\ldots,m$.

It is quickly observed that the information on the parameters γ and σ increases compared to the no Bayesian approach, with $I^{AP}_{\pi}(\gamma,\sigma)>0$; it will obviously reduce the variance / covariance estimation point for the parameters γ et σ given by the Cramer Rao inequality (equation 12) for non biased estimators.

IV. Optimization Problem and SimulationAlgorithm

A. *Principle of proposed methodology*

The objective of methodology is to define an optimal accelerated testing plan considering an economic approach. The objective function is defined by two terms: the cost linked to testing activities and the cost associated to operation of the product. In many works, the optimum testing plans are defined in considering a prior knowledge on reliability parameters (choice of reliability function, scale and shape parameters …) and acceleration model (choice of model, model parameters …) to evaluate the proportions of failure at each accelerated level. This prior knowledge is not integrated in statistical inference during the optimisation process (minimization of reliability metric variance) and the estimation process with testing data. Nevertheless, this information can be used in Bayesian inference (to optimise the testing plan) but it contains the uncertainty on real reliability of new product. So, the proposed methodology consists to define an optimal accelerated testing plan in considering an objective function based on economic approach, Bayesian inference for optimizing the test plan and taking into account the uncertainty on parameters to obtain a robust optimal testing plan.

The figure 3 presents the steps of proposed methodology.

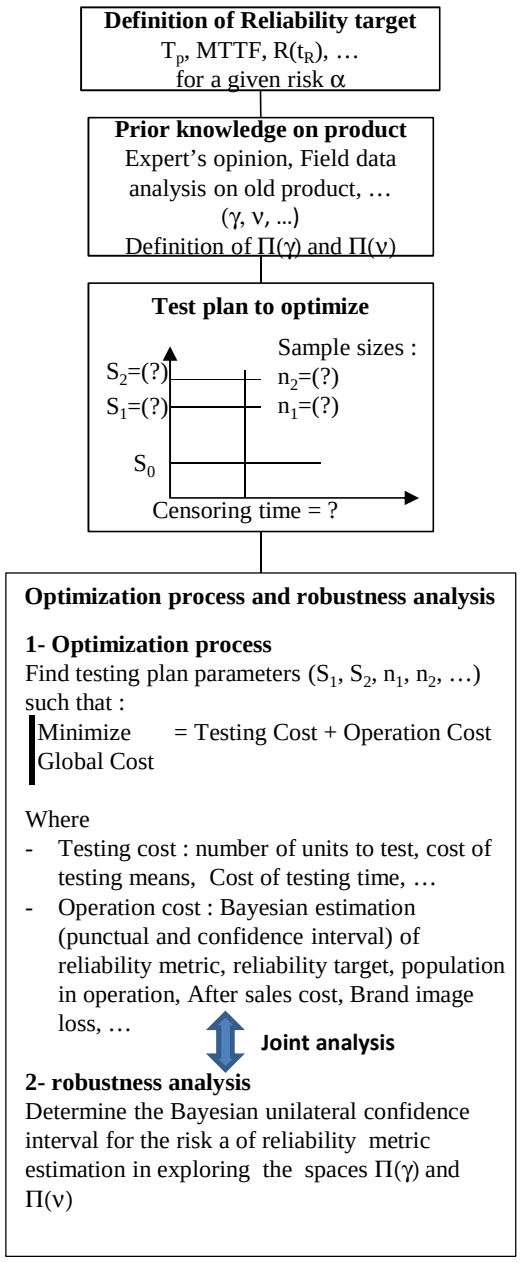


Figure 3. Principle of proposed methodology

In following subsections, the different steps are detailed in considering a simple example to illustrate the methodology.

B. *Definition of reliability target*

The test plan is designed to demonstrate the contractual reliability metric target. Various metrics are used to characterize the reliability of products: MTTF, B_{10}, probability of failure for the warranty period… The verification consists to evaluate the risk to not reach the reliability target with the estimations of punctual estimate and confidence interval. In this paper, the probability of failure $p_{operation}$ for the operation time $t_{operation}$ is considered.

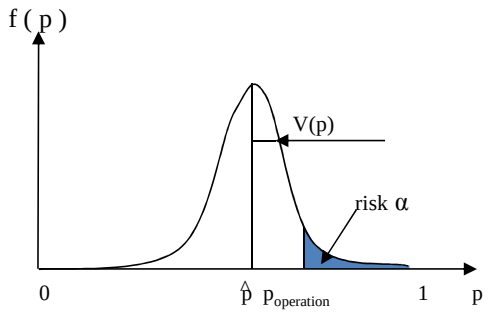


Figure 4. Probability function of operation failure f(p)

The ponctual estimate is defined by

$$\hat{p} = 1 - R_{X_0}\left(t_{operation}\right) = R\left(\left(\ln t_{operation} - \hat{\gamma}^T Z_0\right)\hat{\nu}\right) \qquad (20.)$$

And the variance of probability

$$V(p) \sim \sum_{l=0}^{m}\left(\frac{\partial\left(1 - R_{X_0}(t_{operation})\right)}{\partial \gamma_l}\right)^2_{\hat{\gamma}_l,\hat{\nu}} V(\gamma_l) + \left(\frac{\partial\left(1 - R(t)\right)}{\partial \nu}\right)^2_{\hat{\gamma}_l,\hat{\nu}} V(\nu) \qquad (21.)$$

The parameters $\hat{\gamma}_1, \hat{\upsilon}$, ν, V($\gamma_l$) and V(ν) are obtained by Monte Carlo simulation defined in section IV.F.

The probability of failure in operation period is estimated by the unilateral confidence interval for the risk α (see figure 4). The beta distribution f(p) is considered to characterize the distribution of p. The beta parameters are estimated by the moments method:

$$\alpha_0 = \frac{\hat{p}^2}{v_p}(1-\hat{p}) - \hat{p}$$

$$\beta_0 = \left(\frac{\hat{p}}{v_p}\right)(1-\hat{p})^2 + \hat{p} - 1$$

The probability $p_{operation}$ is deducted from:

$$p_{operation} = \text{Beta}^{-1}(1-\alpha|\alpha_0, \beta_0) \qquad (22.)$$

C. Prior knowledge on product

The prior knowledge on new product ((γ, ν, ...) is obtained from Expert's opinion, Field data analysis on old product, Reliability Standard, …. By the assumption of independence between the variables, the joint distribution can be defined as:

$$\pi(\gamma,\nu) = \pi_\nu(\nu)\left[\prod_{l=0}^{m}\pi_{\gamma_l}(\gamma_l)\right] \qquad (23.)$$

The choice of the form of Π(.) depends on degree of knowledge on parameter (γ_l or ν).

D. Test plan to optimize

In this paper, a simple test plan (with two accelerated constant stress levels) is studied defined as following:

- the censoring time τ is fixed
- the sample size n is fixed
- the stress S_2 is fixed to maximum
- the middle stress level S_1 is unknown
- the proportion allocation p_1 of sample size at stress level S_1 is unknown (n_1= ent(p_1.n) and n_2 = n – n_1)

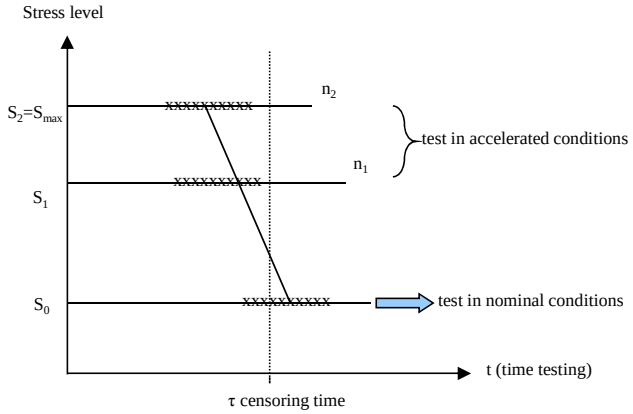


Figure 5. optimal test plan

The decision variables of test plan optimization are:

- the middle stress level S_1
- proportion allocation p_1 of sample size at stress level S_1.

E. Objective Function

The objective of the accelerated testing plan optimization is to minimize the global cost (defined by the costs of testing and operation), which is described as:

$$C_{Global} = C_{testing} + C_{operation} \qquad (24.)$$

where

$C_{testing}$ = n.unit price + fixed testing cost

+ τ.cost per testing hour + cost per batch.(n_1/max units number per batch)$^{\gamma testing}$ + (n_2/max units number per batch)$^{\gamma testing}$

$C_{operation}$ = fixed operation cost +

($P_{operation}$- P_{target}).product population.unit cost

+cost of Brand image loss.($P_{operation}$ – P_{target})$^{\gamma operation}$

F. Optimization procedure

The optimization procedure is simplified in defining the solutions space on prior domain (p_1=]0, 1[, S_1=]S_0, S_1[ ). The prior domain is discretized:

$p_1 = \{p_1\text{min}, \ldots, p_1 i, \ldots, p_1\text{max}\}$ with i = 0, m_1

$S_1 = \{S_1\text{min}, \ldots, S_1 j, \ldots, S_1\text{max}\}$ with j = 0, m_2

For each point (i,j) of discretized domain the global cost is estimated by Monte Carlo simulation:

1- Generate random values of γ_l and ν from prior distribution $\Pi\,\gamma_l(.)$ and $\Pi\,\nu(.)$.

2- Generate random time to failure with respect of censoring time τ at each stress levels with random variables γ_l and ν generated in step 1.

3- Estimate γ_l, ν, $V(\gamma_l)$ and $V(\nu)$ by relationships (17.) and (18.)

4- Estimate the p and V(p) by relationships (20.) and (21.).

5- Repeat $n_{simulation}$ time the steps 1 to 4 (This repetition allows to take into account the sample size effect and the uncertainty on real reliability of new product characterized by the prior distributions.)

6- Estimate probability of operation (eq. 22.) in considering the means of p and V(p) on all repetitions.

7- Evaluate the global cost (i,j)

Finally, the global cost surface is approximated by quadratic polynomial. The optimal test plan parameters are obtained by a classical optimisation method in minimizing the cost described by the quadratic polynomial.

V. NUMERICAL EXAMPLE

In this section, the proposed methodology is applied to ball bearings with a numerical example. We present firstly ball bearing life model [2]. Secondly, we define a simulation model to simulate reliability testing results. Finally, the optimization procedure is detailed. This example is given with numerical data based on the knowledge of ball bearing.

A. Ball Bearings Life

Rolling element bearings are used in an extremely wide tools variety, like machinery and equipment. They may be used in fans, gear boxes, transmissions, axles, compressors, electric motors, engines, final drives, jet engine main shafts, blenders, saws, mixers, etc. Most rotating shafts use a rolling element bearing [2].

Ball bearing life refers to the amount of time a bearing will perform in a specified operation before failure. Bearing life is commonly defined in terms of L_{10} life, which is sometimes referred to as B_{10}. This is the life which 90% of identical bearings subjected to identical usage applications and environments will attain (or surpass) before bearing material fails from fatigue. In practice, life (in million of revolutions) given as a function of load is represented with Palmgren's relationship [3] for the L_{10} of the life distribution:

$$L_{10}=\left(\frac{C}{P}\right)^{p} \qquad (25.)$$

Where C is the basic dynamic load rating,

P is the equivalent radial load,

p is the load life exponent (equal to 3 for ball bearings and 10/3 for rolling bearings).

Bearing life is usually defined by the Weibull distribution, which is combined with the Palmgren's relationship:

$$R(u,\,P)=e^{-0.105\left(\frac{u}{L_{10}}\right)^{\nu}} \qquad (26.)$$

The shape parameter ν (generally equal to 1.5) is used and the scale parameter η is given by:

$$\eta(P)=L_{10}\left(\frac{1}{0.105}\right)^{\frac{1}{\nu}} \qquad (27.)$$

or scale parameter η_H in hour

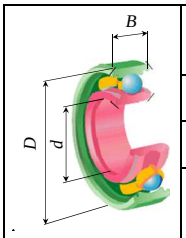
$$\eta_H(P)=L_{10}\frac{60.N}{10^6}\left(\frac{1}{0.105}\right)^{\frac{1}{\nu}} \qquad (1)$$

with N the speed in revolution per minute.

B. Simulation Model definition of testing result

For the optimization, the times to failure are drawings from a Weibull distribution. The ball bearing is characterized by the dimensional parameters given in TABLE I.

TABLE I. SIMULATION DATA

	D = 32 mm : outside diameter
	d = 15 mm : inside diameter
	B = 8 mm : width bearing
	C = 400 daN : Basic dynamic load rating
	N = 1500 tr/min : operating rotation shaft speed

The load P is used to accelerate the test. The simulation parameters are defined in table II.

TABLE II. PRIOR INFORMATION FOR SVA MODEL

C	[350 ; 450]
$\Pi(C)$	Normal(400; 28.86) defined by the moments method
ν	[1.3 ; 1.7]
$\Pi(\nu)$	Normal(1.5; 0.1155) defined by the moments method

The TABLE II. gives the values of prior knowledge on C and ν. Note that, for ball bearing, a shape parameter equal to 1.5 is commonly used based on huge data analysis. According

to that, we consider a prior knowledge following a normal distribution with a low uncertainty.

The accelerated test conditions used to generate random times to failure and censoring times are defined in table III.

TABLE III. PRIOR INFORMATION FOR SVA MODEL

T censoring time	300 h
C_k : random value drawn from $\Pi(C)$	C_k
ν_k : random value drawn from $\Pi(\nu)$	ν_k
S_2 : maximum stress level	175 DaN
S_1 : middle stress level defined by the prior discretized domain (p_1, S_1).	S_1^j
n : total sample size	60
n_1 : sample size at stress S_1 deducted from value p_1 defined by the prior discretized domain (p_1, S_1).	n_1^i
n_2 : sample size at stress S_2 deducted from value n_1 by $n_2 = n - n_1$	n_2^i

The scale parameter η in hour is given by:

$$\eta_i(S_2) = \left(\frac{C_k}{S_2}\right)^3 \frac{60.N}{10^6}\left(\frac{1}{0.105}\right)^{\frac{1}{\nu_k}}$$

$$\eta_i(S_1) = \left(\frac{C_k}{S_1}\right)^3 \frac{60.N}{10^6}\left(\frac{1}{0.105}\right)^{\frac{1}{\nu_k}}$$

The random test times are obtained from Weibull distributions at S_1 and S_2 (defined by the shape parameter ν_k et scale parameters $\eta_k(S_1)$ and $\eta_k(S_2)$) with the respect of censoring time τ and sample sizes n_1 and n_2.

The random test times are analyzed and γ_l, ν, $V(\gamma_l)$ and $V(\nu)$ estimations are obtained (relationships (17.) and (18.)). The estimation of probability of failure p and its variance V(p) are performed using relationships (20.) and (21.).

C. *Optimization procedure*

For this example, the optimization procedure is simplified in evaluating the value of global cost at each point of discretized space of solutions with p_1 and S_1 defined by:

p_1 = {0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95}

S_1 = {125, 130, 135, 140, 145, 150, 155, 160, 165, 170}

For each point (i,j) of discretized domain, the probability of failure is estimated with the simulation method presented in previous section. This step is repeated 500 times to take into account the sample size effect and the uncertainty on real reliability of new product characterized by the prior distributions. The estimation of operation probability (eq. 22.) is obtained in considering the means of p and V(p) on all repetitions. The approach is implemented in MATLAB.

Finally, the cost function (testing and operation) can be obtained. The parameters for evaluating the testing cost (resp. operation cost) are given in TABLE IV. (resp. TABLE V.).

TABLE IV. VALUES OF PARAMETERS FOR EVALUATING THE TESTING COST

n	60 units
Censoring time	300 hours
Testing unit price	500 €
Testing fixed cost	10.000 €
Cost per hour	100 €
Cost per batch	1000 €
Max units number per batch	30
$\gamma_{testing}$	2.6

TABLE V. VALUES OF PARAMETER FOR EVALUATING THE OPERATION COST

P target	0.01
Sales population	1000 units
Warranty time	100 hours
Replace unit price	100 €
After sales cost	10.000 €
Brand image loss	100 €
$\gamma_{operation}$	1.3

The figure 5 represents the testing cost in solutions space (p_1, S_1) in considering the cost defined by eq (23.) and the parameter values given in TABLE IV.

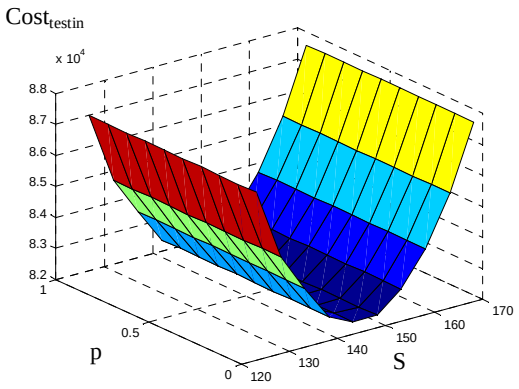


Figure 6. Testing cost in solutions space (p_1, S_1).

The Figure 7. represents the operation cost in solutions space (p_1, S_1) in considering the cost defined by equation 24. and the parameter values given in TABLE II. and TABLE V.

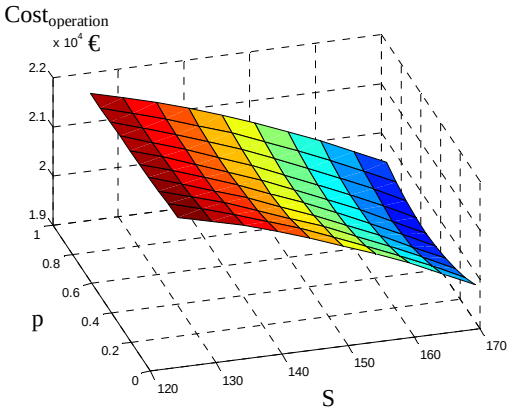


Figure 7. Operation cost in solutions space (p1, S1).

The Figure 8. represents the global cost integrating the Testing and Operation costs.

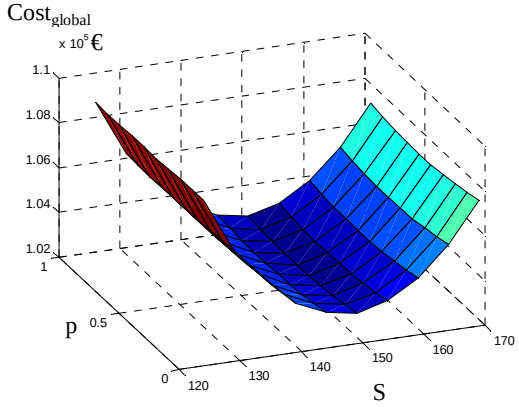


Figure 8. Global cost in solutions space (p_1, S_1).

The Figure 9. represents the Iso-global cost and we can show the area where the optimal testing plan is placed.

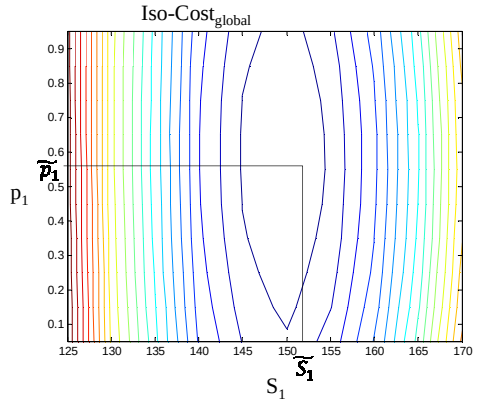


Figure 9. Iso-Global cost in solutions space (p_1, S_1).

The optimal test plan parameters are obtained precisely in fitting the surface by a quadratic polynomial defined by:

$$\text{Cost}_{\text{global}}(p_1, S_1) = a_0 + a_1.p_1 + a_2.S_1 + a_3.p_1^{\,2} + a_4.S_1^{\,2} + a_5.p_1.S_1$$

The TABLE VI. gives the parameter values of fitting quadratic polynomial on global cost as below.

TABLE VI. PARAMETER VALUES OF FITTING QUADRATIC POLYNOMIAL

a_0	119785.123
a_1	-127.372
a_2	-27037.498
a_3	0.420
a_4	25142.345
a_5	-1.082

The optimal test plan parameters are obtained by a classical optimisation method in minimizing the cost described by the quadratic polynomial. The minimum is reached for: p_1 = 0.541 and S_1 = 152.33 with $\text{Cost}_{\text{Global}}$ = 102770 €, $\text{Cost}_{\text{Testing}}$ = 82189 € and $\text{Cost}_{\text{operation}}$ = 20581 €.

Finally, the test plan to launch in order to optimize the global cost is defined by the TABLE VII.

TABLE VII. PARAMETER RESULTS TO OPTIMIZE TEST PLAN

N	60 units
n_1	32 units
n_2	28 units
S_1	152.33 DaN
S_2	175 DaN
Censoring time	300 hours

VI. CONCLUSION

In this paper, we introduced a general framework to obtain optimal accelerate test plans with a cost objective. Then, we proposed a simulation to solve our problem applied to ball bearing. For the sake of simplicity, we restricted our presentation to an elementary optimization problem with two parameters solve by a discretization algorithm.

Future related work will attempt at extending our approach to develop theoretical formulation in various freedom parameters to define the test plan. These development need to improve the algorithm of optimization.

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