Some results on the feedback control of max-plus linear systems under state constraints

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C. A. Maia and L. Hardouin and J. E. R. Cury

***Abstract*— The aim of this paper is to study a control problem for a class of restriction, which appears in design of feedback controllers for timed discrete event systems, taking performance (eigenvalue of the closed loop matrix) and realization issues into account. The obtained results are based on properties of the system and the restriction matrices. For a given class of theses matrices, it is shown that a causal feedback can be computed by solving linear equations. In order to illustrate the applicability of the results, we solve a small traffic light problem.**

I. INTRODUCTION

Max-plus algebra is suitable to deal with a class Timed Discrete Event Systems (TDES) that presents synchronization and delay phenomena. The dynamic model of these systems can be expressed by using only "sum" and "maximization" operations [3]. One important kind of such systems are the max-plus linear systems (MPLS), whose dynamic model can be represented, in a compact form, by using matrices as for the linear conventional system. MPLS can be represented by using Timed Event Graph (TEG), which is a class of timed Petri net in which all places have only one input and only one output transition [27]. Applications of MPLS include modeling and control of computer, transportation and production systems.

Concerning control, in [26] and [19] control strategies are proposed to deal with systems with uncontrollable inputs; Finite horizon control problems for uncertain system are addressed in [28]. In [9] a closed-loop control approach based on transfer function and on reference model has been proposed, which has been extended in [20] in order to take parameter uncertainties into account by using interval analysis. In [21], another transfer function approach is developed for multivariable control. Further generalization of model reference control based on transfer function is presented in [25] and [24].

This paper deals with the constraint control of max-plus linear systems. Classes of this problem can be approached by using directly the system realization, based on daters (or counters), like in [2], [13], [18], [22], [23] or its transformed version based on power series, which is similar to Z-transform in conventional linear system theory [30], [29], [17]. Unlike previous paper, we are interested in a class of problems described by using a state space realization of the system, as in [18], [22], [23], however our problem formulation is different in the sense that we do not need to restrict the initial condition to be in a given semimodule. In Addition, we can take into account performance (eigenvalue of the closed loop matrix) and realization issues in the feedback synthesis. The presented results is obtained by means of the system and the restriction matrices. We show that a causal feedback can be computed by solving linear equations for a given class of these matrices. The applicability of the results is illustrated by solving a small traffic light problem.

The paper organization is as follows. Section II introduces some algebraic tools concerning max-plus algebra, residuation theory and max-plus linear equations. Section III presents the control problem and the main results. Numerical results for a small traffic problem are shown in Section IV. A conclusion is given in Section V.

II. ALGEBRAIC FRAMEWORK

In this paper all models are described by means of a dioid. A dioid is an algebraic structure defined by a set $\mathbb{D}$ with the operations $\oplus$ and $\otimes$, which is denoted by $(\mathbb{D}, \oplus, \otimes)$. The operation $\oplus$ is associative, commutative and idempotent, that is, $a \oplus a = a, \forall a \in \mathbb{D}$, and has neutral element denoted by ε. The operation $\otimes$ is associative and distributive on the left and on the right with respect to $\oplus$ and has neutral element denoted by e. Moreover, for all a, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$, that is, ε is absorbing with respect to $\otimes$. In a dioid, a partial order relation is defined by $b \preceq a$ iff $a = a \oplus b$. As a consequence, unlike conventional algebra based on real field, operation $\otimes$ is isotone, that is, it does not change the order.

A dioid $\mathbb{D}$ is said to be complete if it is closed under infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. In some situations the symbol $\otimes$ will be omitted as in conventional algebra, that is $a \otimes b = ab$. The i^{th} power of an element a in a dioid is defined as in conventional case, that is $a^i = a \otimes a^{i-1}$ and $a^0 = e$.

In this paper, all the models are expressed in terms of the dioid $\mathbb{Z}_{\max}$, which denotes the dioid $(\mathbb{Z} \cup \{-\infty\}, max, +)$. We recall that this dioid has neutral elements that can be interpreted as by $\varepsilon = -\infty$ and $e = 0$. Hereafter we denote by I the max-plus identity matrix, defined analogously as the conventional case, considering the neutral elements of $\mathbb{Z}_{\max}$.

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The same is done for a zero matrix. Matrix operations are defined analogously as the conventional case, as well.

A. Residuation theory

Residuation theory [4] is useful to deal with many control problems described by dioids. This theory deals with the conditions for the existence of the greatest element x for the inequality $f(x) \preceq y$ in partially ordered sets.

In this paper we are interested in the mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ defined over a complete dioid $\mathbb{D}$. It can be proved that they are both residuated [3]. Their residuals are isotone mappings denoted respectively by $L_a^\sharp(y) = a \backslash\!\!\circ\, y$ and $R_a^\sharp(y) = y \,\circ\!\!/\, a$. Dually, if there exists a least element x for the inequality $y \preceq f(x)$ it is denoted by $f^\flat(y)$. Mapping $f^\flat$ is called the *dual residual* of f. For instance, the mapping $T_a(x) = x \oplus a$, defined over a complete dioid $\mathbb{D}$, is dually residuated, and its residual is denoted by $T_a^\flat(y) = y \ominus a$.

Particularly, we are interested in matrix inequalities. For inequality matrices of the type $AX \preceq B$ in the dioid $\mathbb{D}^{n\times n}$, there exists a greatest solution $A \backslash\!\!\circ\, B$, which the entries can be computed by:

$$(A \backslash\!\!\circ\, B)_{ij} = \bigwedge_{l=1}^{n} A_{li} \backslash\!\!\circ\, B_{lj},$$

in which $X \wedge Y$ stands for the greatest element lower than or equal to X and Y.

B. Semimodule and max-plus equations

A semimodule is equivalent to the notion of linear vector space in a semiring setting. A semimodule defined from a dioid $(\mathcal{D}, \oplus, \otimes, \varepsilon_s, e)$ is a comutative monoid $(\mathcal{M}, \widehat{\oplus})$ with neutral element $\varepsilon_\mathcal{M}$, equipped with a map $(\mathcal{D} \times \mathcal{M}) \mapsto \mathcal{M}$, that is $(\lambda, v) \mapsto \lambda.v$ (left action), for which:

$$\begin{aligned}
(\lambda \otimes \mu).v &= \lambda.(\mu.v),\\
\lambda.(u \widehat{\oplus} v) &= \lambda.u \widehat{\oplus} \lambda.v,\\
(\lambda \oplus \mu).v &= \lambda.v \widehat{\oplus} \mu.v,\\
\varepsilon_s.v &= \varepsilon_\mathcal{M},\\
\lambda.\varepsilon_\mathcal{M} &= \varepsilon_\mathcal{M},\\
e.v &= v,
\end{aligned}$$

for all $u, v \in \mathcal{M}$ and $\lambda, \mu \in \mathcal{D}$. For more details see [14], [8].

In [5], [14], [1] it has been shown that the set of all solutions of the system $Ax = Bx$, for which A, B, x have entries in $\mathbb{Z}_{\max}$, can be characterized by a finitely generated semimodule. More precisely it can be expressed as an image of a matrix with entries in $\mathbb{Z}_{\max}$. In addition, the system $Ax = Bx$ is equivalent to $\bar{A}z = \bar{B}y$.

Remark 1: The equation $Ax = Bx$ is equivalent to $\bar{A}z = \bar{B}y$ with

$$\bar{A} = \begin{bmatrix} A \\ I \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ I \end{bmatrix}$$

in which I is an identity matrix, y and z are vectors, all with appropriate dimension

Therefore if the issue is to find only one solution for the linear system $Ay = Bz$, we can use the "alternating" algorithm proposed by [11] (and extended to dioids of intervals by [16]). Assuming that A and B have at least one nonzero element on each row and on each column, and initializing with a matrix without ε entries, the authors show that the algorithm converges if and only if a nonzero solution exists. More recently, in order to deal with complexity issues to solve these equations, [12] have developed another algorithm that uses dynamic programming and dynamical games techniques. This algorithm is based on fixed point algorithms developed in [15], [6]. In this paper, the algorithm of [11] will be used in the computation of a feedback control law, however the one proposed by [12] is an another possibility to be investigated in future works.

In this context, if a finite solution for this linear system exists, the following algorithm, which was adapted from [11], can provide a solution in finite number of step.

Begin
Choose arbitrary finite element z
$y := A \backslash\!\!\circ\, (Bz)$;
While $(Ay \neq Bz)$,

$$\begin{aligned}
z &:= B \backslash\!\!\circ\, (Ay);\\
y &:= A \backslash\!\!\circ\, (Bz);
\end{aligned}$$

End
End

Remark 2: We remark that there is no solution if any line of matrix A is strictly greater than the respective line in matrix B and vice-versa and this condition should be tested before running the algorithm.

Remark 3: The following example presents an easy way to compute a particular semimodule which appears in many max-plus problems.

$$Ex \preceq x \quad \Leftrightarrow \quad E^*x = x \Leftrightarrow x \in \mathsf{Im}E^*, \tag{1}$$

in which E is a matrix and $E^* = \bigoplus_{i\in\mathbb{N}} E^i \quad ((.)^*$ is called "Kleene star operator"). See [3] for more details.

In addition, since we are dealing with control synthesis, we must ensure that the control law is realizable. A linear feedback control law is realizable if the feedback matrix is causal. In this sense, the following definition is useful.

Definition 1 (Causal Matrix): A matrix $M \in (\mathbb{Z}_{\max})^{n\times p}$ is said to be causal if its entries are such that $M_{ij} = \varepsilon$ or $M_{ij} \succeq e$.

III. Control Synthesis

In a general way, the state evolution of a MPLS can be described by the following system:

$$x(k) \quad = \quad Ax(k-1) \oplus Bu(k), \tag{2}$$

in which vectors $x(k) \in (\mathbb{Z}_{\max})^n$ and $u(k) \in (\mathbb{Z}_{\max})^p$ represent respectively the date of k^{th} firing of the internal (state) and input transitions, A and B are the system matrices of appropriate dimensions. We recall that, since $x(k) \preceq x(k+1)$, that is, the firing dates are nondecreasing, then

$I \preceq A$, in which I is the identity matrix. Max-plus linear systems can be handled by using toolboxes for Scilab and a C^{++} library [7], [10].

Definition 2 (Control problem): The aim is to find a realizable feedback control law, $u(k) = Fx(k-m)$, for the following max-plus linear system:

$$x(k) = Ax(k-1) \oplus Bu(k), \tag{3}$$

to ensure that the state will evolve in order to be kept in the following semimodule:

$$Dx(k) = x(k), \quad \forall k \geq k_0 \tag{4}$$

for which $F \in (\mathbb{Z}_{\max})^{p\times n}$, $D \in (\mathbb{Z}_{\max})^{n\times n}$ and $m, k_0 > 0$, whatever be the initial condition $x(0)$ of the system.

Remark 4: We remark that this control problem, is a generalization of the control problem presented in [18], [22], since here the initial state does not necessarily belong to a given semimodule. In other words, the main concern here is to design a control law in order to drive the state of system into the semimodule $Dx = x$ and keep it inside.

If there exists a linear feedback control of the form $u(k) = Fx(k-1)$ with $F \in (\mathbb{Z}_{\max})^{p\times n}$, hence Eq. 3 becomes:

$$x(k) = (A \oplus BF)x(k-1), \quad \forall k \geq 1. \tag{5}$$

As a consequence, provided that the state evolution of the system is given by Eq.5, a solution exists if and only if there exists $k_0 \geq 1$ such that:

$$D(A \oplus BF)x(k-1) = (A \oplus BF)x(k-1), \quad \forall k \geq k_0. \tag{6}$$

Therefore, we can obtain conditions ensuring the existence and computation of a solution. If the following equation holds:

$$D(A \oplus BF) = A \oplus BF, \tag{7}$$

a solution exists whatever be the initial condition. And as a consequence, the following lemma will be useful to find a solution.

Lemma 1: Any matrix F that respects $A \preceq BF$ and $DBF = BF$ is a solution of Eq.7.

Proof: Since $A \preceq BF$ then $A \oplus BF = BF$. Therefore $D(A \oplus BF) = DBF$ and $(A \oplus BF) = BF$. Equation $DBF = BF$ completes the proof. ■

As we can see, we have to deal with equation of the type $Az = Bz$. In this sense, the following definition is useful (see [11].

Definition 3 (Row (column) G-astic Matrix): A matrix is row (column) G-astic if it has at least one nonzero in each row (column).

Lemma 2 ([22]): If B is row G-astic[1], $\forall L \in \mathbb{Z}^{n\times n}_{max}$ it is possible to choose a matrix M, by making its elements large enough, such that $L \preceq BM$.

[1]We can observe that if B is row G-astic, each state of the system has a connection to at least one input.

Definition 4 (Matrix construction from a vector): We denote by $Z^{(z)} \in \mathbb{Z}^{p\times n}$ a matrix in which columns are identical and formed by the vector $z \in \mathbb{Z}^p$. We denote by $F^{(z,\alpha)}$ a matrix such that $F^{(z,\alpha)} = Z^{(z)}\alpha$ in which $\alpha \in \mathbb{Z}^{n\times n}$ is a diagonal matrix.

By using this definition we have following lemma.

Lemma 3: If a vector z is such that $DBz = Bz$, then the same equation holds for any $F^{(z,\alpha)}$, that is, $DBF^{(z,\alpha)} = BF^{(z,\alpha)}$.

As a consequence we can state the following proposition.

Proposition 1: If the matrix B is row G-astic and if there exists a vector $z \in \mathbb{Z}^p_{max}$, with non null entries such that $DBz = Bz$, there exists a solution for Eq.7.

Proof: If z is such that $(z_i \neq \varepsilon)$, and B is row G-astic, then by Def.3, any $F^{(z,\alpha)}$ is such that such that $DBF^{(z,\alpha)} = BF^{(z,\alpha)}$. By Lem.2, it is always possible to make $A \preceq BF^{(z,\alpha)}$, since we can make the matrix $F^{(z,\alpha)}$ as large as needed. In this sense, if we denote $F = F^{(z,\alpha)}$, we can ensure that $DBF = BF$ and $A \preceq BF$. Therefore by Lem.1, F is a solution for Eq.7. ■

Remark 5: It is straightforward to show that the least α such that $A \preceq BF^{(z,\alpha)}$ can be easily computed and it is equal to $\alpha_{min} = -diag\{A \backslash\!\!\!\circ (BZ^{(z)})\}$, in which $diag\{X\}$ indicates the diagonal of matrix X. Therefore if the requirements of Proposition 1 are fulfilled, all F such that $F \succeq Z^{(z)}\alpha_{min}$ are solutions for Eq. 7.

A. Dealing with realization and performance issues

Once the requirements of proposition 1 are fulfilled we can obtain a solution for the control problem. However, the feedback can increase the eigenvalue of the closed-loop matrix $A_f = A \oplus BF$ and so the firing dates of the state transition can be delayed. This can be an issue in some applications. In order to deal with this issue, we can investigate what we can do with non realizable solution since they can lead to smaller eigenvalues for matrix $A_f = A \oplus BF$. In this sense one question is: given a non realizable solution F for Eq. 7 is it possible to use it in order to design another realizable solution for the problem? The answer is yes if matrix A_f is irreducible and has eigenvalue bigger than 0. We will discuss this point in the following.

If there exists a non realizable control law $u_{nc}(k) = F_{nc}x(k-1)$ (by this we mean that matrix F_{nc} is non causal) then

$$x(k) = (A \oplus BF_{nc})x(k-1), \tag{8}$$

and for a given $k_0 \geq 1$, we have

$$D(A \oplus BF_{nc})x(k-1) = (A \oplus BF_{nc})x(k-1), \quad \forall k \geq k_0. \tag{9}$$

Given any initial condition $x(0)$, by using Eq. 8, the control law can be rewritten as:

$$u_{nc}(k) = F_{nc}(A \oplus BF_{nc})^{m-1}x(k-m), \quad \forall k \geq m, \tag{10}$$

provided that

$$\begin{aligned} x(1) &= (A \oplus BF_{nc})x(0) \\ &\vdots \\ x(m) &= (A \oplus BF_{nc})x(m-1) \end{aligned}$$

From the spectral theory of max-plus matrices, we can show for any irreducible matrix H that

$$H^{k+c} = (\lambda)^c H^k, \forall k \geq p \tag{11}$$

for p large enough, in which λ is the eigenvalue and c is the cyclicity of matrix H. Please see [3] for more details. In this sense, since $A \oplus BF_{nc} \succeq I$ and provided that matrix $A \oplus BF_{nc}$ is an irreducible matrix with eigenvalue greater than 0 (which is not too restrictive), we can always have a causal matrix $F = F_{nc}(A \oplus BF_{nc})^{m-1}$ by increasing m as much as it needed[2]. From a feedback realization in terms of a TEG, this means that we have places with m tokens and sojourn times given by the entries of matrix F. This ideas will be illustrated in the example given in the next section.

IV. Illustrative example: a small traffic light problem

Consider the TEG depicted in Fig.1 as an example. It

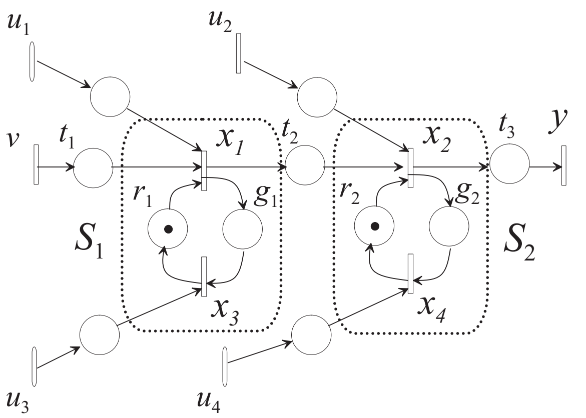

Fig. 1. A Small Traffic Light System

may describe a road (a green wave) with two traffic lights, indicated by S_1 and S_2. Transitions x_i with $i \in \{1,2\}$ indicates an event when the semaphore i turns green light on and transitions x_i, $i \in \{3,4\}$ indicate when red lights turn on. r_i is the minimal duration of the red time for the semaphore, g_i the duration of the green time, and u_i a control input which can increase the red and green times. In this kind of model the flow of cars is represented by tokens crossing the graph from transition v until to transition y. Each of these tokens are assumed to represent a virtual platoon of cars (see for instance [13] for a presentation of this concept). Virtual means that a token represents the presence of cars even when the cars aren't there, that is, in this model the traffic light system runs independently of the car flow since we do not assume sensors of car presence on the road. In other words, input v is not a constraint and it is assumed to be fired an infinite number of time, which is equivalent to consider $v = \varepsilon$.

The size of the platoon on each section is related to the green duration g_i which is fixed a priori, it is assumed to be sufficiently small to ensure that the platoon may be physically contained in the downstream section. The time duration t_i represents the minimal time allowing to a platoon to go from one location to other, e.g. t_2 is the time necessary for the platoon to leave the first traffic light and to reach the second one. The max-plus model of this system (with $v = \varepsilon$) is given by:

$$x(k) = A_0 x(k) \oplus A_1 x(k-1) \oplus B_0 u(k). \tag{12}$$

in which[3]:

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ t_2 & \varepsilon & \varepsilon & \varepsilon \\ g_1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & g_2 & \varepsilon & \varepsilon \end{bmatrix}, A_1 = \begin{bmatrix} e & \varepsilon & r1 & \varepsilon \\ \varepsilon & e & \varepsilon & r_2 \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix},$$

and,

$$B_0 = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}.$$

Since A_0 has circuits with negative circuit weights this equation can be rewritten as (see [3]):

$$x(k) = A_0^* A_1 x(k-1) \oplus A_0^* B_0 u(k). \tag{13}$$

Therefore the matrix A for the system is $A_0^* A_1$ and the matrix B is $A_0^* B_0$. We remark that the matrix B is row G-astic.

In this system we aim to ensure the following time synchronization constraints:

- The maximum sojourn in the place between x_1 and x_2 is T_2, that is $x_2(k) - x_1(k) \leq T_2$. In other words the platoons have to wait less than $T_2 - t_2$ in the front of the second traffic light.
- The maximum green time of the first and the second semaphore is respectively G_1 and G_2, that is $x_3(k) - x_1(k) \leq G_1$ and $x_4(k) - x_2(k) \leq G_2$

These constraints lead to an inequality of the form $E_r x(k) \preceq x(k)$, in which E_r is given by:

$$E_r = \begin{bmatrix} \varepsilon & -T_2 & -G_1 & \varepsilon \\ \varepsilon & \varepsilon & -G_2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

Furthermore by Eq.12, the state must also respect the inequality $A_0 x(k) \preceq x(k)$. So $(E_r \oplus A_0)x(k) \preceq x(k)$. As a consequence, by using Equiv. 1 the system must enforces

$$Dx(k) = x(k) \tag{14}$$

[2]We remark that this procedure ensure that m is always finite

[3]Diagonal entries of A_1 are equal to e to enforce that firing dates are not decreasing, that is $x(k) \succeq x(k-1)$.

in which $D = (E_r \oplus A_0)^*$.

In order to present numerical results, we chose for the system: $t_2 = 10$, $g_1 = 4$, $g_2 = 5$, $r_1 = 5$, $r_2 = 7$. And for the constraints: $G_1 = G2 = 15$ and $T_2 = 15$. These leads to the following matrices for the system:

$$A = \begin{bmatrix} 0 & \varepsilon & 5 & \varepsilon \\ 10 & 0 & 15 & 7 \\ 4 & \varepsilon & 9 & \varepsilon \\ 15 & 5 & 20 & 12 \end{bmatrix}, B = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ 10 & 0 & \varepsilon & \varepsilon \\ 4 & \varepsilon & 0 & \varepsilon \\ 15 & 5 & \varepsilon & 0 \end{bmatrix},$$

and for the constraints:

$$D = \begin{bmatrix} 0 & -15 & -15 & -30 \\ 10 & 0 & -5 & -15 \\ 6 & -11 & 0 & -26 \\ 15 & 5 & 0 & 0 \end{bmatrix}.$$

By solving the equation $DBz = Bz$, with the algorithm proposed by [11], we have

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

As a result, following results Sec. III we can find solutions $F^{(z)}$ such that $D(A \oplus BF^{(z)}) = A \oplus BF^{(z)}$ for the proposed control problem. A particular interesting one is the one that does not change (or cause the least change in) the greatest eigenvalue of matrix A. Following Rem. 5, we can show that $F_{nc} = [z \quad z \quad z \quad z]\alpha_{min}$, in which $\alpha_{min} = -diag\{A \backslash (BZ^{(z)}))\} = diag[0 \quad -10 \quad 5 \quad -3]$, that is

$$F_{nc} = \begin{bmatrix} 0 & -10 & 5 & -3 \\ 0 & -10 & 5 & -3 \\ 0 & -10 & 5 & -3 \\ 0 & -10 & 5 & -3 \end{bmatrix},$$

does not change the greatest eigenvalue of matrix A.

Following the discussion presented in Subsec. III-A, it is possible to find a feedback control law $u(k) = Fu(k-2)$ in which:

$$F = F_{nc}(A \oplus BF_{nc}) = \begin{bmatrix} 12 & 2 & 17 & 9 \\ 12 & 2 & 17 & 9 \\ 12 & 2 & 17 & 9 \\ 12 & 2 & 17 & 9 \end{bmatrix},$$

with initial conditions given by:
$x(0) = [0 \quad 0 \quad 0 \quad 0 \quad]^T$,
$x(1) = (A \oplus BF_{nc})x(0) = [5 \quad 15 \quad 9 \quad 20 \quad]^T$,

We can observe that we can start the state of system wherever we want, in this example $Dx(0) \neq x(0)$, and in one iteration it will be inside the desired semimodule of constraint. In addition we remark that the control law is realizable in term of a TEG by initially adding appropriate place in the feedback with 2 tokens and respective temporization given by the entries of matrix F. Moreover we also observe that the open-loop response does not ensure $Dx(k) = x(k) \quad \forall k > 1$. For instance, the open-loop response leads to $x(3) = A^3x(0) = [23 \quad 39 \quad 27 \quad 44 \quad]^T$, for which $Dx(3) \neq x(3)$. On the other hand the closed-loop one leads to $x(3) = Ax(2) \oplus BFx(1) = [29 \quad 39 \quad 33 \quad 44 \quad]^T$, that is, the feedback delays appropriately the transition firings in order to keep the state inside the constraint semimodule.

Remark 6: We have presented a simple and easy to understand problem, but with no trivial solution. However the approach is scalable to more complex problems with higher dimension matrices. Its computational complexity relies mainly on the solution of the equation $DBz = Bz$, which is pseudo-polynomial [11].

V. Conclusion

This paper has presented an approach to design a feedback controller in order to enforce a class of constraints for max-plus linear systems. The approach has taken into account initial condition, realization and performances issues in the designing process. The applicability of the results was illustrate by solving a small traffic light problem. This is an on-going work and interesting topic for future research includes investigation on more general conditions concerning the solvability of proposed control problem.

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