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SEMI-ALGEBRAIC FUNCTIONS WITH NON-COMPACT CRITICAL SET

NICOLAS DUTERTRE AND JUAN ANTONIO MOYA PÉREZ

Abstract. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set, $F : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ semi-algebraic function and $f = F_{|X} : X \to \mathbb{R}^n$ be the restriction of F to X. We define the global index of a critical value c_i of f and prove an index formula for $\chi(X)$ that generalizes a result previously proved by the authors for the case of isolated critical points. We define also new indices at infinity and prove an alternative index formula for $\chi(X)$.

1. Introduction

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ with an isolated critical point at 0. The Khimshiashvili formula (see [9]) states that

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - \text{sign}(-\delta)^n \deg_0 \nabla f,$$

where $0 < |\delta| \ll \epsilon \ll 1$, B_ϵ is the closed ball of radius ϵ centered at 0, ∇f is the gradient of f and $\deg_0 \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{|\nabla f|} : S_\epsilon \to S^{n-1}$.

As a corollary of the Khimshiashvili formula, by a result of Arnol'd [1] and Wall [15] we have that

$$\chi(\{f \leq 0\} \cap S_\epsilon) = 1 - \deg_0 \nabla f,$$

$$\chi(\{f \geq 0\} \cap S_\epsilon) = 1 + (-1)^{n-1} \deg_0 \nabla f,$$

and

$$\chi(\{f = 0\} \cap S_\epsilon) = 2 - 2 \deg_0 \nabla f,$$

if n is even.

Szafraniec [11] generalized the results of Arnol'd and Wall to the case of a function germ f with non-isolated singularities and in [12] he improved this result for a weighted homogeneous polynomial $f : \mathbb{R}^n \to \mathbb{R}$.

In [6] Lemma 2.5, the first named author proves a new relation between the topology of the positive (resp. negative) real Milnor fibre of an analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and the topology of the link of the set $\{f \leq 0\}$ (resp. $\{f \geq 0\}$). Using Szafraniec's results, he deduces a generalization of the Khimshiashvili formula for non-isolated singularities. Namely he proves that if $0 < \delta \ll \epsilon$, then

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = 1 - (-1)^n \deg_0 \nabla g_-,$$

and

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - (-1)^n \deg_0 \nabla g_+,$$

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with $g_- = -f - \omega^d$, $g_+ = f - \omega^d$, $\omega(x) = x_1^2 + \cdots + x_n^2$ and d is an integer big enough.

Sekalski in [10] gives a global counterpart of Khimshiasvili's formula for a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ with a finite number of critical points. He considers the set $\Lambda_f = \{\lambda_1, \ldots, \lambda_k\}$ of critical values of f at infinity, where $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, and its complement $\mathbb{R} \setminus \Lambda_f = \cup_{i=0}^k]\lambda_i, \lambda_{i+1}[$ where $\lambda_0 = -\infty$ and $\lambda_{k+1} = +\infty$. Denoting by $r_\infty(g)$ the number of real branches at infinity of a curve $\{g = 0\}$ in $\mathbb{R}^2$, he proves that

$$\deg_\infty \nabla f = 1 + \sum_{i=1}^k r_\infty(f - \lambda_i) - \sum_{i=0}^k r_\infty(f - \lambda_i^+),$$

where for $i = 0, \ldots, k$, λ_i^+ is an element of $]\lambda_i, \lambda_{i+1}[$ and $\deg_\infty \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{\|\nabla f\|} : S_R \to S^{n-1}$, $R \gg 1$.

Gwoździewicz in [7] gives a topological proof of Sekalski's result using Euler integration. He proves that

$$\deg_\infty \nabla f = 1 + \int_{\mathbb{R}} r_\infty(f - t) d\chi_c(t),$$

where χ_c denotes the Euler characteristic with compact support that we will define later.

The first named author generalizes Sekalski result in [3] by considering a closed semi-algebraic set $X \subset \mathbb{R}^n$ and a $\mathcal{C}^2$ semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f_{|X}$ has a finite number of critical points. In [5] the authors recover the first named author's results using Euler integration, which clearly simplifies the proofs.

Finally, in [4], Section 3, Araujo, Chen, Andrade and the first named author gave a generalization of the results of [3] when $X = \mathbb{R}^n$ and f is a semi-tame function with non-isolated critical points, by adapting to the global case the method developed by Szafraniec in [11].

The aim of this paper is to extend these results to the general case, i.e., without any assumption on the set of critical points of the function. We work in the following setting: $X \subset \mathbb{R}^n$ is a closed semi-algebraic set, $F : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$ semi-algebraic function and $f = F_{|X} : X \to \mathbb{R}^n$ is the restriction of F to X.

In Section 3, we define the global index of a critical value c_i of f,

$$\mathrm{ind}_g(f, X, f^{-1}(c_i)) = \chi(f^{-1}(c_i)) - \chi(f^{-1}(c_i - \alpha) \cap B_{R_{c_i}}),$$

where $R_{c_i} \gg 1$ and $0 < \alpha \ll \frac{1}{R_{c_i}}$. Then, we generalize Theorem 3.16 of [3] and Theorem 5.1 of [5] for the case of a non-compact critical set, that is, we prove that (Theorem 3.2)

$$\chi(X) = \sum_{i=1}^l \mathrm{ind}_g(f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \le t\})) d\chi_c(t).$$

Using the same techniques we generalize the other results of [3] and [5] for $\chi(X)$ for the case of a non-compact critical set. As an application, we obtain an index formula for the quotient of two semi-algebraic functions.

In Section 4 we define two new indices, the right index at infinity of an asymptotic non-ρ-regular value d_i (see Definition 2.14),

$$\mathrm{ind}_\infty^+(f, X, f^{-1}(d_i)) = \chi(f^{-1}(d_i + \alpha)) - \chi(f^{-1}(d_i + \alpha) \cap B_{R_{d_i}}),$$

and the left index at infinity of d_i,

$$\mathrm{ind}_\infty^-(f, X, f^{-1}(d_i)) = \chi(f^{-1}(d_i - \alpha)) - \chi(f^{-1}(d_i - \alpha) \cap B_{R_{d_i}}),$$

where $R_{d_i} \gg 1$ and $0 < \alpha \ll \frac{1}{R_{d_i}}$. We compute these indices in particular cases and we finish the section with a formula that relates $\chi(X)$ with them (Theorem 4.4).

In Section 5 we generalize Corollaries 3.28, 3.29, 3.30 of [4] to our general situation. Namely, we construct two functions g_+ and g_- that have compact sets of critical points and then we prove that the sum of the global indices of f (respectively $-f$) and g_- (respectively g_+) coincide (Corollary 5.11).

2. Some preliminary results

2.1. **Euler integration.** Let $X \subset \mathbb{R}^n$ be a semi-algebraic set. We can write it in the following way:

$$X = \sqcup_{j=1}^{l} C_j,$$

where C_j is semi-algebraically homeomorphic to $]-1,1[^{d_j}$ (C_j is called a cell of dimension d_j). We set

$$\chi_c(X) = \sum_{j=1}^{l} (-1)^{d_j},$$

and we call it the Euler characteristic with compact support of X. Let us remark that if X is compact, then $\chi_c(X) = \chi(X)$.

A constructible function $\varphi : X \to \mathbb{Z}$ is a $\mathbb{Z}$-valued function that can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i 1_{X_i},$$

where X_i is a semi-algebraic subset of X.

If φ is a constructible function, the Euler integral of φ is defined as

$$\int_X \varphi d\chi_c(x) = \sum_{i \in I} m_i \chi_c(X_i).$$

Definition 2.1. Let $f : X \to Y$ be a continuous semi-algebraic map and let $\varphi : X \to \mathbb{Z}$ be a constructible function. The push forward $f_*\varphi$ of φ along f is the function $f_*\varphi : Y \to \mathbb{Z}$ defined by

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c(x).$$

Theorem 2.2. *(Fubini's theorem) Let $f : X \to Y$ be a continuous semi-algebraic map and let φ be a constructible function on X. Then, we have*

$$\int_Y f_*\varphi d\chi_c(y) = \int_X \varphi d\chi_c(x).$$

Corollary 2.3. *Let X, Y be semi-algebraic sets and let $f : X \to Y$ be a continuous semi-algebraic map. Then*

$$\chi_c(X) = \int_Y \chi_c(f^{-1}(y)) d\chi_c(y).$$

2.2. **Stratified critical points and values.** Let us consider from now on a closed semi-algebraic set $X \subset \mathbb{R}^n$. It is equipped with a finite semi-algebraic Whitney stratification $X = \sqcup_{a \in A} S_a$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$-semi-algebraic function and let $f = F_{|X}$.

Definition 2.4.

(1) A point $p \in X$ is a critical point of f if it is a critical point of $F_{|S(p)}$, where $S(p)$ is the stratum that contains p.

(2) A point $c \in \mathbb{R}$ is a critical value if there exists $p \in f^{-1}(c)$ such that p is a critical point of f.
(3) If p is an isolated critical point of f, we define the index of f at p by

$$\operatorname{ind}(f, X, p) = 1 - \chi(\{f = f(p) - \delta\} \cap B_\epsilon(p)),$$

where $0 < \delta \ll \epsilon \ll 1$.

Let us notice that if $X = \mathbb{R}^n$, by [9], $\operatorname{ind}(f, X, p) = \deg_p \nabla f$.

Lemma 2.5. *The set of critical points of f, Σ_f, is a closed semi-algebraic subset of X and its set of critical values, Δ_f, is finite.*

Proof. To prove that Σ_f is closed we use Whitney's condition (a) and to prove that Δ_f is finite we use the Bertini-Sard Theorem ([2]). □

The following result gives a relation between the Euler characteristic of X and the indices of the p_i's, when X is compact.

Theorem 2.6. *([3], Theorem 3.1) If X is compact and f has a finite number of critical points $p_1, \dots, p_l$, we have*

$$\chi(X) = \sum_{i=1}^{l} \operatorname{ind}(f, X, p_i).$$

Now, we give some lemmas that we will use later on. For the proofs we refer to [3]. We assume that f has a finite number of critical points $p_1, p_2, \dots, p_l$.

Lemma 2.7. *If $\delta < 0$ is a small regular value of f and $R \gg 1$ is such that $f^{-1}(0) \cap B_R$ is a retract by deformation of $f^{-1}(0)$, then*

$$\chi(f^{-1}(\delta) \cap B_R) = \chi(f^{-1}(0)) - \sum_{p_i \in f^{-1}(0)} \operatorname{ind}(f, X, p_i).$$

Lemma 2.8. *If f is proper then for any $\alpha \in \mathbb{R}$, we have*

$$\chi(\{f \geq \alpha\}) - \chi(\{f = \alpha\}) = \sum_{i : f(p_i) > \alpha} \operatorname{ind}(f, X, p_i).$$

The following lemma is a consequence of Lemma 2.8 and the Mayer-Vietoris sequence.

Lemma 2.9. *If f is proper then for α and α' with $\alpha < \alpha'$, we have*

$$\chi(\{\alpha \leq f \leq \alpha'\}) - \chi(\{f = \alpha\}) = \sum_{i : \alpha < f(p_i) \leq \alpha'} \operatorname{ind}(f, X, p_i).$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$-semi-algebraic function such that $g^{-1}(0)$ intersects X transversally. Let us suppose that $f_{|X \cap \{g \leq 0\}}$ admits an isolated critical point p in $X \cap \{g = 0\}$ which is not a critical point of f. We say that such a point is a correct critical point. If S denotes the stratum of X that contains p, this implies that

$$\nabla(f_{|S})(p) = \lambda(p) \nabla(g_{|S})(p),$$

with $\lambda(p) \neq 0$.

Lemma 2.10. *For $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi(f^{-1}(-\delta) \cap B_\epsilon(p) \cap X \cap \{g \leq 0\}) = 1,$$

if $\lambda(p) > 0$ and

$$\chi(f^{-1}(-\delta) \cap B_\epsilon(p) \cap X \cap \{g \leq 0\}) = \chi(f^{-1}(-\delta) \cap B_\epsilon(p) \cap X \cap \{g = 0\}),$$

if $\lambda(p) < 0$.

Remark 2.11. As a consequence of the last lemma and the definition of the index of a critical point p, we get that

$$\mathrm{ind}(f, X \cap \{g \leq 0\}, p) = 0,$$

if $\lambda(p) > 0$, and

$$\mathrm{ind}(f, X \cap \{g \leq 0\}, p) = \mathrm{ind}(f, X \cap \{g = 0\}, p),$$

if $\lambda(p) < 0$.

2.3. **Link at infinity and adapted radius.** For any closed semi-algebraic set equipped with a Whitney stratification $X = \sqcup_{\alpha \in A} S_\alpha$, we denote by $\mathrm{Lk}^\infty(X)$ the link at infinity of X. It is defined as follows. Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a C^2 proper semi-algebraic positive function. Since $\omega_{|X}$ is proper, the set of critical points of $\omega_{|X}$ (in the stratified sense) is compact. Hence for R sufficiently big, the map $\omega : X \cap \omega^{-1}([R, +\infty[) \to \mathbb{R}$ is a stratified submersion. The link at infinity of X is the fibre of this submersion. The topological type of $\mathrm{Lk}^\infty(X)$ does not depend on the choice of the function ω (for instance, see [3], Section 3).

Definition 2.12. We will say that $R > 0$ is an adapted radius for X if $D : X \cap D^{-1}([R, +\infty[) \to \mathbb{R}$ is a stratified submersion, where D is the euclidean norm.

Remark 2.13. We note that if R is an adapted radius for X then $\mathrm{Lk}^\infty(X)$ is homeomorphic to $X \cap S_{R'}$, for $R' \geq R$.

2.4. **Asymptotic non-ρ-regular values.** Let $\rho(x) = 1 + \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. Note that $\nabla\rho(x) = x$, $\rho(x) \geq 1$ and the levels of ρ are the spheres of radius greater than or equal to 1. Let $\Gamma_{f,\rho}$ be the polar set

$$\Gamma_{f,\rho} = \left\{ x \in \mathbb{R}^n \mid \mathrm{rank}[\nabla f_{|S}(x), \nabla \rho_{|S}(x)] < 2 \right\},$$

where S is the stratum that contains x. We have $\Sigma_f \subset \Gamma_{f,\rho}$.

Definition 2.14. The set of asymptotic non-ρ-regular values of f is the set defined as follows:

$$\Lambda_f = \{\alpha \in \mathbb{R} \mid \exists \{x_n\}_{n \in \mathbb{N}} \in \Gamma_{f,\rho} \text{ such that } |x_n| \to +\infty \text{ and } f(x_n) \to \alpha\}.$$

The set Λ_f was introduced and studied by Tibăr [14] when $X = \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial. By Lemma 2.2 in [3], we can assume that $\Gamma_{f,\rho} \setminus \Sigma(f)$ is a curve and so, that Λ_f is a finite set $\{d_1, d_2 \ldots, d_m\}$.

2.5. **Some others sets of special values.** We define four sets of special values. They are values where some changes in the topology of the fibres of f may occur.

Definition 2.15. Let $* \in \{\leq, =, \geq\}$.

(1) We define Λ_f^* by

$$\Lambda_f^* = \{\alpha \in \mathbb{R} \mid \beta \mapsto \chi(\mathrm{Lk}^\infty(\{f * \beta\})) \text{ is not constant in a neighborhood of } \alpha\}.$$

(2) We define $\widetilde{B}(f)$ by $\widetilde{B}(f) = \Delta_f \cup \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$.

Proposition 2.16.

(1) *The sets Λ_f^* and $\widetilde{B}(f)$ are finite. Moreover $\Lambda_f^{=} \subset \Lambda_f^{\leq} \cup \Lambda_f^{\geq}$.*

(2) *If $\alpha \notin \widetilde{B}(f)$, the functions*

$$\beta \mapsto \chi(\{f * \beta\}), * \in \{\leq, =, \geq\},$$

are constant in a neighborhood of α.

Proof. The first point is proved in [3]. Let $\alpha \notin \widetilde{B}(f)$ and let $\alpha^- < \alpha$ be a value close enough to α. Let R_α (resp. R_{α^-}) be an adapted radius for $f^{-1}(\alpha)$ (resp. $f^{-1}(\alpha^-)$). We can choose them in such a way that they are also adapted to $\{f \le \alpha\}$ and $\{f \le \alpha^-\}$ respectively. The critical points of $f_{|\{\alpha^- < f < \alpha\} \cap B_{R_{\alpha^-}}}$ can only lie on $S_{R_{\alpha^-}}$, and they point outwards. By Lemma 2.9, this implies that

$$\chi(\{f \le \alpha^-\}) = \chi(\{f \le \alpha\}),$$

because R_{α^-} is also adapted for $\{f \le \alpha\}$.

Similarly, we can consider the critical points of $-f_{|\{\alpha^- < f < \alpha\} \cap B_{R_{\alpha^-}}}$. Applying Lemma 2.9 twice, we obtain that

$$\chi(\{f \ge \alpha^-\}) - \chi(\{f \ge \alpha\}) = \chi(\mathrm{Lk}^\infty(\{f \ge \alpha^-\})) - \chi(\mathrm{Lk}^\infty(\{f \ge \alpha\})) = 0,$$

since $\alpha \notin \Lambda_f^{\ge}$. By the Mayer-Vietoris sequence, we see that

$$\chi(\{f = \alpha^-\}) = \chi(\{f = \alpha\}).$$

The same proof works for $\alpha^+ > \alpha$, a value close enough to α. □

Remark 2.17. Taking into account Proposition 2.16 and basic properties of χ_c, if we have inclusions

$$\Lambda_f^* \subset \{c_1, c_2, \dots, c_s\},$$

with $c_1 < c_2 < \cdots < c_s$ and

$$\widetilde{B}(f) \subset \{d_1, d_2, \dots, d_k\},$$

with $d_1 < d_2 < \cdots < d_k$, we can express the Euler integral $\int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(X \cap \{f * t\})) d\chi_c(t)$ as

$$\int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(X \cap \{f * t\})) d\chi_c(t) = \sum_{i=1}^{s} \chi(\mathrm{Lk}^\infty(X \cap \{f * c_i\})) - \sum_{i=0}^{s} \chi(\mathrm{Lk}^\infty(X \cap \{f * c_i^+\})),$$

the Euler integral $\int_{\mathbb{R}} \chi(X \cap \{f * t\}) d\chi_c(t)$ as

$$\int_{\mathbb{R}} \chi(X \cap \{f * t\}) d\chi_c(t) = \sum_{j=1}^{k} \chi(X \cap \{f * d_j\}) - \sum_{j=0}^{k} \chi(X \cap \{f * d_j^+\}),$$

and the Euler integral $\int_{\mathbb{R}} \chi_c(X \cap \{f * t\}) d\chi_c(t)$ as

$$\int_{\mathbb{R}} \chi_c(X \cap \{f * t\}) d\chi_c(t) = \sum_{j=1}^{k} \chi_c(X \cap \{f * d_j\}) - \sum_{j=0}^{k} \chi_c(X \cap \{f * d_j^+\}),$$

where $c_0, d_0 = -\infty$, $c_{s+1} = +\infty$, $d_{k+1} = +\infty$, $c_i^+ \in]c_i, c_{i+1}[$ and $d_j^+ \in]d_j, d_{j+1}[$.

3. Formulas for the Euler characteristic of a closed semi-algebraic set in the general case

Let X be a closed semi-algebraic set, equipped with a finite semi-algebraic Whitney stratification $X = \sqcup_{a \in A} S_a$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ semi-algebraic function. We call $f = F_{|X}$, the restriction of F to X. Let $\Delta(f) = \{c_1, c_2, \dots, c_k\}$ be the set of critical values of f.

Let c_i be a critical value of f. The partition $f^{-1}(c_i) = \sqcup_{a \in A} f^{-1}(c_i) \cap S_a$ may not be a Whitney stratification, but since Whitney conditions are stratifying, we can refine it in order to get a Whitney stratification $f^{-1}(c_i) = \sqcup_{b \in B} T_B$ of $f^{-1}(c_i)$ such that

$$X = \sqcup_{a \in A} (S_a \setminus f^{-1}(c_i)) \bigsqcup \sqcup_{b \in B} T_B$$

is still a Whitney stratification of X.

Definition 3.1. We define the index of a critical value c_i of f as

$$\mathrm{ind}_g(f, X, f^{-1}(c_i)) = \chi(f^{-1}(c_i)) - \chi(f^{-1}(c_i - \alpha) \cap B_{R_{c_i}})$$

with $0 < \alpha \ll 1$ and R_{c_i} is an adapted radius for $f^{-1}(c_i)$.

Theorem 3.2. *We have*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}_g(f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^{\infty}(\{f \leq t\}))d\chi_c(t).$$

Proof. By Hardt's theorem [8], there exists a finite set $\widetilde{\Delta}_f \subset \mathbb{R}$ such that over each connected component of $\mathbb{R} \setminus \widetilde{\Delta}_f$, f is a semi-algebraic trivial fibration. Let us write

$$\Lambda_f \cup \widetilde{B}_f \cup \widetilde{\Delta}_f = \{b_1, \ldots, b_l\},$$

where $b_1 < \cdots < b_l$.

Note that, by Lemma 2.7, $\mathrm{ind}_g(f, X, f^{-1}(b_j)) = 0$ if $b_j \notin \Delta(f)$.

By Corollary 2.3, we have

$$\chi_c(X) = \int_{\mathbb{R}} \chi_c(f^{-1}(t))d\chi_c(t) = \sum_{j=1}^{l}(\chi_c(f^{-1}(b_j) - \chi_c(f^{-1}(b_j^-)) - \chi_c(f^{-1}(b_l^+)),$$

where $b_j^- = b_j - \alpha$ and $b_j^+ = b_j + \alpha$, with $0 < \alpha \ll 1$.

To compute the right-hand side of the above equality, we work with each difference $\chi_c(f^{-1}(b_j) - \chi_c(f^{-1}(b_j^-))$ for $j = 1, \ldots, l$. Let us set $b_j^- = b^-$ and $b_j = b$ with $b^- = b - \delta$, $0 < \delta \ll \frac{1}{R_{b^-}}$, where $R_{b^-} > R_b \gg 1$ are adapted radius for $f^{-1}(b^-)$ and $f^{-1}(b)$. We have

$$\begin{aligned} &\chi_c(f^{-1}(b)) - \chi_c(f^{-1}(b^-)) \\ &= \chi(f^{-1}(b)) - \chi(\mathrm{Lk}^{\infty}(f^{-1}(b)) - \chi(f^{-1}(b^-) \cap B_{R_b}) + \chi(f^{-1}(b^-) \cap S_{R_b}) \\ &\quad -\chi_c(f^{-1}(b^-) \cap \{|x| \geq R_b\}) \\ &= \mathrm{ind}_g(f, X, f^{-1}(b)) - \chi(\mathrm{Lk}^{\infty}(f^{-1}(b)) + \chi(f^{-1}(b^-) \cap S_{R_b}) - \chi_c(f^{-1}(b^-) \cap \{|x| \geq R_b\}). \end{aligned}$$

As explained above, we can assume that $f^{-1}(b)$ is a union of strata of our stratification. If R_b is sufficiently big and b^- is sufficiently close to b, then the (stratified) critical points of $-\rho_{|\{b^- \leq f \leq b\}}$ lying in $\{R_b \leq \rho \leq R_{b^-}\}$ appear on $\{f = b^-\}$. Moreover they are correct and points outwards (Figure 1).

Therefore, by Lemmas 2.9 and 2.10, we have

$$\begin{aligned} &\chi(\{R_b \leq |x| \leq R_{b^-}\} \cap \{b^- \leq f \leq b\}) \\ &= \chi(\{b^- \leq f \leq b\} \cap S_{R_{b^-}}) = \chi(\{f \leq b\} \cap S_{R_{b^-}}) - \chi(\{f \leq b^-\} \cap S_{R_{b^-}}) + \chi(\{f = b^-\} \cap S_{R_{b^-}}) \\ &= \chi(\mathrm{Lk}^{\infty}(\{f \leq b\})) - \chi(\mathrm{Lk}^{\infty}(\{f \leq b^-\})) + \chi(\mathrm{Lk}^{\infty}(\{f = b^-\})), \end{aligned}$$

applying the Mayer-Vietoris sequence and the definition of the link at infinity.

Let us compute $\chi(\{R_b \leq |x| \leq R_{b^-}\} \cap \{b^- \leq f \leq b\})$ in another way. Let $\tilde{b}$ be a regular value of f such that $b^- < \tilde{b} < b$ and $f_{|\{R_b \leq |x| \leq R_{b^-}\}}$ has no critical point on $\{\tilde{b} \leq f < b\}$. This implies that $f^{-1}(b) \cap \{R_b \leq |x| \leq R_{b^-}\}$ is a deformation retract of $\{R_b \leq |x| \leq R_{b^-}\} \cap \{\tilde{b} \leq f \leq b\}$. Applying the same argument as above, considering the function $f_{|\{R_b \leq |x| \leq R_{b^-}\}}$ and applying Lemmas 2.9 and 2.10, we obtain that

$$\chi(\{b^- \leq f \leq \tilde{b}\} \cap \{R_b \leq |x| \leq R_{b^-}\}) = \chi(\{f = b^-\} \cap \{R_b \leq |x| \leq R_{b^-}\}).$$

By the Mayer-Vietoris sequence and the deformation retract argument, we get that

$$\begin{aligned} \chi(\{b^- \leq f \leq b\} \cap \{R_b \leq |x| \leq R_{b^-}\}) &= \chi(\{f = b^-\} \cap \{R_b \leq |x| \leq R_{b^-}\}) \\ &+\chi(\{f = b\} \cap \{R_b \leq |x| \leq R_{b^-}\}) - \chi(\{f = \tilde{b}\} \cap \{R_b \leq |x| \leq R_{b^-}\}). \end{aligned}$$

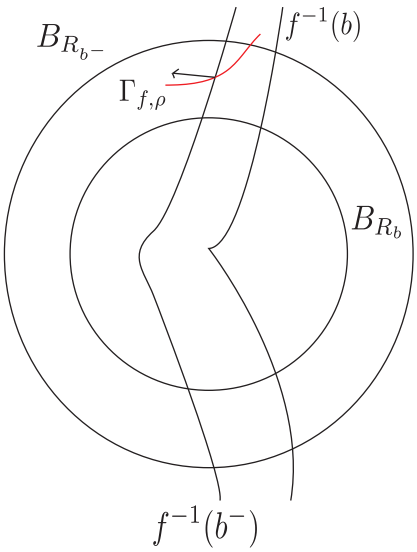

FIGURE 1

Moreover if we choose $\tilde{b}$ close enough to b, then the intersection
$$\Gamma_{f,\rho} \setminus \Sigma_f \cap [f^{-1}([\tilde{b}, b]) \cap \{R_b \le |x| \le R_{b^-}\}]$$
is empty (see Figure 2).

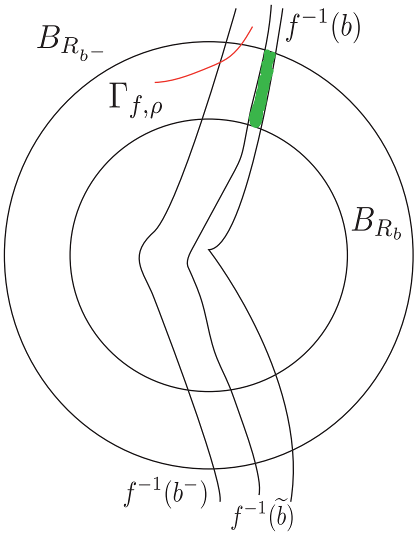

FIGURE 2

This implies that
$$\chi(\{f = \tilde{b}\} \cap \{R_b \le |x| \le R_{b^-}\}) = \chi(\{f = \tilde{b}\} \cap S_{R_b}).$$
Finally we obtain that
$$\begin{aligned}\chi(\{b^- \le f \le b\} \cap \{R_b \le |x| \le R_{b^-}\}) &= \chi(\{f = b^-\} \cap \{R_b \le |x| \le R_{b^-}\}) \\ &+\chi(\mathrm{Lk}^\infty(f^{-1}(b))) - \chi(\{f = b'\} \cap S_{R_b}).\end{aligned}$$

Comparing the two expressions for $\chi(\{R_b \leq |x| \leq R_{b^-}\} \cap \{b^- \leq f \leq b\})$ leads to

$$\chi(\{f = b^-\} \cap \{R_b \leq |x| \leq R_{b^-}) = \chi(\mathrm{Lk}^\infty(\{f \leq b\})) - \chi(\mathrm{Lk}^\infty(\{f \leq b^-\}))$$
$$+\chi(\mathrm{Lk}^\infty(\{f = b^-\})) - \chi(\mathrm{Lk}^\infty(f^{-1}(b))) + \chi(\{f = b'\} \cap S_{R_b}).$$

Then we can write

$$\chi_c(\{f = b^-\} \cap \{|x| \geq R_b\}) = \chi(\{f = b^-\} \cap \{|x| \geq R_b\}) - \chi(\mathrm{Lk}^\infty(f^{-1}(b^-)))$$
$$= \chi(\{f = b^-\} \cap \{R_b \leq |x| \leq R_{b^-}\}) - \chi(\mathrm{Lk}^\infty(f^{-1}(b^-)))$$
$$= \chi(\mathrm{Lk}^\infty(\{f \leq b\})) - \chi(\mathrm{Lk}^\infty(\{f \leq b^-\})) - \chi(\mathrm{Lk}^\infty(f^{-1}(b))) + \chi(\{f = b'\} \cap S_{R_b}).$$

Finally we obtain

$$\chi_c(f^{-1}(b)) - \chi_c(f^{-1}(b^-)) = \mathrm{ind}_g(f, X, f^{-1}(b))$$
$$-\chi(\mathrm{Lk}^\infty(\{f \leq b\})) + \chi(\mathrm{Lk}^\infty(\{f \leq b^-\})),$$

and so

$$\chi_c(X) = \sum_{j=1}^{l} \mathrm{ind}_g(f, X, f^{-1}(b_j)) - \sum_{i=1}^{l} \chi(\mathrm{Lk}^\infty(\{f \leq b_j\}))$$
$$+\sum_{i=1}^{l} \chi(\mathrm{Lk}^\infty(\{f \leq b_j^-\})) - \chi_c(f^{-1}(b_l^+)).$$

Let $R_{b_l^+}$ be an adapted radius for $f^{-1}(b_l^+)$. We can write

$$\chi_c(f^{-1}(b_l^+)) = \chi(f^{-1}(b_l^+) \cap B_{R_{b_l^+}}) - \chi(\mathrm{Lk}^\infty(f^{-1}(b_l^+)))$$
$$= \chi(\{f \geq b_l^+\} \cap B_{R_{b_l^+}}) - \chi(\mathrm{Lk}^\infty(f^{-1}(b_l^+))),$$

because by Lemma 2.8, $\chi(f^{-1}(b_l^+) \cap B_{R_{b_l^+}}) = \chi(\{f \geq b_l^+\} \cap B_{R_{b_l^+}})$. Hence,

$$\chi_c(f^{-1}(b_l^+)) = \chi(\{f \geq b_l^+\}) - \chi(\mathrm{Lk}^\infty(f^{-1}(b_l^+))) =$$
$$\chi_c(\{f \geq b_l^+\}) + \chi(\mathrm{Lk}^\infty(\{f \geq b_l^+\})) - \chi(\mathrm{Lk}^\infty(f^{-1}(b_l^+))).$$

But since $\chi_c([b_{l^+}, +\infty[) = 0$ and $f_{|[b_{l^+}, +\infty[}$ is a trivial fibration, we get that $\chi_c(\{f \geq b_l^+\}) = 0$. We conclude that

$$\chi_c(f^{-1}(b_l^+)) = \chi(\mathrm{Lk}^\infty(X)) - \chi(\mathrm{Lk}^\infty(\{f \leq b_l^+\})),$$

by the Mayer-Vietoris sequence.

Putting $b_l^+ = b_{l+1}^-$, where $b_{l+1} = +\infty$, we have

$$\chi_c(X) = \chi(X) - \chi(\mathrm{Lk}^\infty(X)) =$$
$$\sum_{j=1}^{l} \mathrm{ind}_g(f, X, f^{-1}(b_j)) - \chi(\mathrm{Lk}^\infty(X)) + \sum_{i=1}^{l+1} \chi(\mathrm{Lk}^\infty(\{f \leq b_j^-\})) - \sum_{i=1}^{l} \chi(\mathrm{Lk}^\infty(\{f \leq b_j\})) =$$
$$\sum_{j=1}^{l} \mathrm{ind}_g(f, X, f^{-1}(b_j)) - \chi(\mathrm{Lk}^\infty(X)) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \leq t\})) d\chi_c(t),$$

obtaining the desired result.

□

Corollary 3.3. *If f has a finite number of critical points $p_1, p_2, \ldots, p_l$ then*

$$\chi(X) = \sum_{i=1}^{l} \mathrm{ind}(f, X, p_i) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \leq t\})) d\chi_c(t).$$

Proof. Let b_i be a critical value such that $f^{-1}(b_i)$ has a finite number of singularities $p_1, \dots, p_{r_i}$. By Lemma 2.7, we know that

$$\sum_{i=1}^{r_i} \operatorname{ind}(f, X, p_i) = \chi(f^{-1}(b_i)) - \chi(f^{-1}(b_i^-) \cap B_{R_{b_i}})$$

where R_{b_i} is an adapted radius for $f^{-1}(b_i)$. □

Corollary 3.4. *We have*

$$\chi(X) = \sum_{i=1}^{k} \operatorname{ind}_g(-f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^{\infty}(\{f \geq t\}))d\chi_c(t).$$

Proof. By replacing f by $-f$ and applying an analogous procedure as in the last theorem, we arrive to the desired result. □

Corollary 3.5. *We have*

$$\begin{aligned} &2\chi(X) - \chi(\mathrm{Lk}^{\infty}(X)) \\ &= \sum_{i=1}^{l} \operatorname{ind}_g(f, X, f^{-1}(c_i)) + \sum_{i=1}^{l} \operatorname{ind}_g(-f, X, f^{-1}(c_i)) \\ &\quad - \int_{\mathbb{R}} \chi(\mathrm{Lk}^{\infty}(\{f = t\}))d\chi_c(t). \end{aligned}$$

Proof. It follows from Theorem 3.2 and Corollary 3.4 by applying the Mayer-Vietoris sequence. □

Lemma 3.6. *We have* $\int_{\mathbb{R}} \chi_c(\{f \leq t\})d\chi_c(t) = 0$.

Proof. Let us take b in $\Lambda_f \cup \widetilde{B}_f \cup \widetilde{\Delta}_f$ and $b^+ = b + \delta$, with $\delta > 0$ small enough, a regular value. Since $f_{|X \cap]b, b^+]}$ is trivial and $\chi_c(]b, b^+]) = 0$, we conclude that

$$\chi_c(\{f \leq b^+\}) - \chi_c(\{f \leq b\}) = \chi_c(\{\alpha < f \leq b^+\}) = 0.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \chi_c(\{f \leq t\})d\chi_c(t) &= \sum_{j=1}^{l} \chi_c(\{f \leq b_i\}) - \sum_{j=0}^{l} \chi_c(\{f \leq b_i^+\}) \\ &= -\chi_c(\{f \leq b_0^+\}) = 0. \end{aligned}$$

□

Corollary 3.7. *We have*

$$\chi(X) = \sum_{i=1}^{k} \operatorname{ind}_g(f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\{f \leq t\})d\chi_c(t).$$

Proof. We have

$$\chi(X) = \sum_{i=1}^{k} \operatorname{ind}_g(f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\mathrm{Lk}^{\infty}(\{f \leq t\}))d\chi_c(t),$$

and

$$\begin{aligned} \int_{\mathbb{R}} \chi_c(\{f \leq t\})d\chi_c(t) &= \int_{\mathbb{R}} \chi_c(\{f \leq t\} \cap B_{R_t})d\chi_c(t) \\ &\quad - \int_{\mathbb{R}} \chi_c(\mathrm{Lk}^{\infty}(\{f \leq t\}))d\chi_c(t) = 0. \end{aligned}$$

Then,

$$\int_{\mathbb{R}} \chi(\{f \leq t\})d\chi_c(t) = \int_{\mathbb{R}} \chi(\{f \leq t\} \cap B_{R_t})d\chi_c(t)$$

$$= \int_{\mathbb{R}} \chi_c(\{f \leq t\} \cap B_{R_t}) d\chi_c(t) = \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \leq t\})) d\chi_c(t),$$

arriving to the desired result. □

Corollary 3.8. *We have*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}_g(-f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\{f \geq t\}) d\chi_c(t).$$

Proof. By replacing f by $-f$ and applying an analogous procedure as in the last corollary, we arrive to the desired result. □

Corollary 3.9. *We have*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}_g(f, X, f^{-1}(c_i)) + \sum_{i=1}^{k} \mathrm{ind}_g(-f, X, f^{-1}(c_i)) - \int_{\mathbb{R}} \chi(\{f = t\}) d\chi_c(t).$$

Proof. It follows from the last two corollaries by applying the Mayer-Vietoris sequence. □

Remark 3.10. Since $\mathrm{ind}_g(f, X, f^{-1}(t)) = 0$ if t is not a critical value of f, we can replace $\sum_{i=1}^{k} \mathrm{ind}_g(\pm f, X, f^{-1}(c_i))$ with $\int_{\mathbb{R}} \mathrm{ind}_g(\pm f, X, f^{-1}(t)) d\chi_c(t)$ in all our statements.

Application 3.11. Let us apply these results to the case of a function given as the quotient of two semi-algebraic functions. Let $f, g : X \to \mathbb{R}$ be two semi-algebraic functions, where X a closed semi-algebraic set and f (resp. g) is the restriction to X of a $\mathcal{C}^2$ semi-algebraic function F (resp. G). We consider their quotient $\phi := f/g : X \setminus V(g) \to \mathbb{R}$ which is also a semi-algebraic function. Let Y be the following closed semi-algebraic set:

$$Y = \{(x, y) \in X \times \mathbb{R} \mid f(x) - yg(x) = 0\}.$$

We cannot apply Corollary 3.9 since ϕ is not defined in X, so we work with Y to obtain a formula for the sum of the global indices of the function ϕ.

Let $y : Y \to \mathbb{R}$ be the linear function defined by $y(x, y) = y$. By applying Corollary 3.9, we have that

$$\chi(Y) = \int_{\mathbb{R}} \mathrm{ind}_g(y, Y, y^{-1}(t)) d\chi_c(t) + \int_{\mathbb{R}} \mathrm{ind}_g(-y, Y, y^{-1}(t)) d\chi_c(t)$$
$$- \int_{\mathbb{R}} \chi(Y \cap \{y = t\}) d\chi_c(t).$$

We have that, if $t \neq 0$,

$$Y \cap \{y = t\} = \{(x, t) \mid f(x) - tg(x) = 0\} = \{x \mid \phi(x) = t\} \sqcup \{f = g = 0\},$$

and so,

$$\chi(Y \cap \{y = t\}) = \chi(\{\phi(x) = t\}) + \chi(\{f = g = 0\}).$$

When $t = 0$, we have that

$$Y \cap \{y = 0\} = \{x \mid f(x) = 0\},$$

and so,

$$\chi(Y \cap \{y = 0\}) = \chi(\{f = 0\}).$$

Let us study the global index of y at the non-zero critical value t. We recall that

$$\mathrm{ind}_g(y, Y, y^{-1}(t)) = \chi(Y \cap y^{-1}(t)) - \chi(Y \cap y^{-1}(t - \alpha) \cap B_{R_t}),$$

where R_t is an adapted radius for $y^{-1}(t)$ and $0 < \alpha \ll \frac{1}{R_t}$.

We have

$$(x, t) \in Y \cap y^{-1}(t) \Leftrightarrow f(x) - tg(x) = 0 \Leftrightarrow$$

$$\begin{cases} \phi(x) = t & \text{if } g(x) \neq 0, \\ f(x) = 0 & \text{if } g(x) = 0, \end{cases}$$

then,

$$\chi(Y \cap y^{-1}(t)) = \chi(\{\phi = t\}) + \chi(\{f = g = 0\}).$$

Let us study $\chi(Y \cap y^{-1}(t - \alpha) \cap B_{R_t})$. We have

$$(x, t-\alpha) \in Y \cap y^{-1}(t) \cap B_{R_t} \Leftrightarrow \begin{cases} f(x) - (y_0 - \alpha)g(x) = 0 \\ |(x, t-\alpha)| \leq R_t \end{cases} \Leftrightarrow$$

$$\begin{cases} \phi(x) = t - \alpha, |x| \leq \sqrt{R_t^2 - (t-\alpha)^2} & \text{if } g(x) \neq 0 \\ f(x) = 0 & \text{if } g(x) = 0 \end{cases}.$$

If R_t is big enough and α small enough, then $\widetilde{R} = \sqrt{R_t^2 - (t-\alpha)^2}$ is an adapted radius for $\{\phi = t\}$ and $\{f = g = 0\}$. Therefore we have

$$\chi(Y \cap y^{-1}(t) \cap B_{R_t}) = \chi(\phi = t - \alpha) \cap B_{\widetilde{R}}) + \chi(\{f = g = 0\} \cap B_{\widetilde{R}}).$$

Finally, we get

$$\chi(Y) = \int_{\mathbb{R}^*} \mathrm{ind}_g(\phi, X, \phi^{-1}(t)) d\chi_c(t) + \int_{\mathbb{R}^*} \mathrm{ind}_g(-\phi, X, \phi^{-1}(t)) d\chi_c(t) +$$
$$\mathrm{ind}_g(y, Y, y^{-1}(0)) + \mathrm{ind}_g(-y, Y, y^{-1}(0)) - \chi(\{f = 0\})$$
$$+2\chi(\{f = g = 0\}) - \int_{\mathbb{R}^*} \chi(X \cap \{\phi = t\}) dt.$$

Let us suppose that $X = \mathbb{R}^n$ and that $\{f = 0\}$ and $\{g = 0\}$ intersect transversally and that $\{f = 0\}$ is regular. This implies that Y is smooth and 0 is a regular value of y. In this case, we get

$$\chi(Y) = \int_{\mathbb{R}^*} \mathrm{ind}_g(\phi, X, \phi^{-1}(t)) d\chi_c(t) + \int_{\mathbb{R}^*} \mathrm{ind}_g(-\phi, X, \phi^{-1}(t)) d\chi_c(t)$$
$$-\chi(\{f = 0\}) + 2\chi\{(f = g = 0\}) - \int_{\mathbb{R}^*} \chi(\{\phi = t\}) dt.$$

4. New indices at infinity

We recall that Λ_f is defined by

$$\Lambda_f = \{\alpha \in \mathbb{R} \mid \exists (x_n)_{n \in \mathbb{N}} \in \Gamma_f \text{ such that } |x_n| \to +\infty \text{ and } f(x_n) \to \alpha\},$$

and that it is a finite set $\{d_1, d_2 \ldots, d_m\}$.

Definition 4.1. We define the right index at infinity of d_i as

$$\mathrm{ind}_\infty^+(f, X, f^{-1}(d_i)) = \chi(f^{-1}(d_i^+)) - \chi(f^{-1}(d_i^+) \cap B_{R_{d_i}}).$$

Analogously, we define the left index at infinity of d_i as

$$\mathrm{ind}_\infty^-(f, X, f^{-1}(d_i)) = \chi(f^{-1}(d_i^-)) - \chi(f^{-1}(d_i^-) \cap B_{R_{d_i}}),$$

where $d_i^+ = d_i + \alpha$, $d_i^- = d_i - \alpha$ with $0 < \alpha \ll 1$ and R_{d_i} is an adapted radius for $f^{-1}(d_i)$.

Example 4.2. Let us consider the Broughton polynomial $f(x, y) = y(xy - 1)$ defined on $X = \mathbb{R}^2$.

We have that $\Lambda_f = \{0\}$ and

$$\mathrm{ind}_\infty^+(f, \mathbb{R}^2, f^{-1}(0)) = \chi(f^{-1}(\delta)) - \chi(f^{-1}(\delta) \cap B_{R_0}) = 2 - 3 = -1,$$
$$\mathrm{ind}_\infty^-(f, \mathbb{R}^2, f^{-1}(0)) = \chi(f^{-1}(-\delta)) - \chi(f^{-1}(-\delta) \cap B_{R_0}) = 2 - 3 = -1,$$

with R_0 an adapted radius for 0 and $0 < \delta \ll 1$ (see Figure 3).

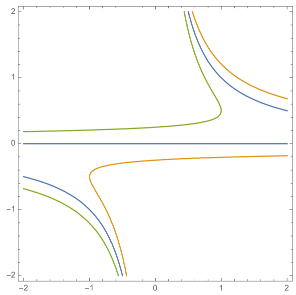

FIGURE 3

Example 4.3. (Tibăr-Zaharia, [13]) Let us consider the polynomial $f(x,y) = x^2y^2 + 2xy + (y^2-1)^2$ defined on $X = \mathbb{R}^2$. We have that $0 \in \Lambda_f$ and

$$\mathrm{ind}^+_\infty(f, \mathbb{R}^2, f^{-1}(0)) = \chi(f^{-1}(\delta)) - \chi(f^{-1}(\delta) \cap B_{R_0}) = 2 - 2 = 0,$$
$$\mathrm{ind}^-_\infty(f, \mathbb{R}^2, f^{-1}(0)) = \chi(f^{-1}(-\delta)) - \chi(f^{-1}(-\delta) \cap B_{R_0}) = 0 - 0 = 0,$$

with R_0 an adapted radius for 0 and $0 < \delta \ll 1$ (see Figure 4).

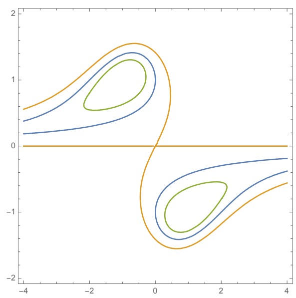

FIGURE 4

By Proposition 2.16, there exists a finite set $\{e_1, e_2, \dots, e_s\}$ such that the function $t \mapsto \chi(f^{-1}(t))$ is locally constant on $\mathbb{R} \setminus \{e_1, e_2, \dots, e_s\}$.

Theorem 4.4. *We have*

$$\chi(X) = \sum_{i=1}^{m} (\mathrm{ind}^+_\infty(f, X, f^{-1}(d_i)) + \mathrm{ind}^-_\infty(f, X, f^{-1}(d_i))) + \int_{[e_1, e_s]} \chi(f^{-1}(t)) d\chi_c(t).$$

Proof. We recall that

$$\Lambda_f \cup \widetilde{B}_f \cup \widetilde{\Delta}_f = \{b_1, \dots, b_l\},$$

First of all, note that, by the definition of the indices at infinity,

$$\mathrm{ind}^+_\infty(f, X, f^{-1}(b_i)) = \mathrm{ind}^-_\infty(f, X, f^{-1}(b_i)) = 0,$$

if $b_i \notin \Lambda(f)$ and that

$$\mathrm{ind}_g(-f, X, f^{-1}(b_i)) = \mathrm{ind}_g(f, X, f^{-1}(b_i)) = 0,$$

if $b_i \notin \Delta(f)$.

By Corollary 3.9, we have

$$\chi(X) = \sum_{j=1}^{l} \mathrm{ind}_g(f, X, f^{-1}(b_i)) + \sum_{j=1}^{l} \mathrm{ind}_g(-f, X, f^{-1}(b_i)) - \int_{\mathbb{R}} \chi(\{f = t\}) d\chi_c(t).$$

By definition,

$$\mathrm{ind}_g(f,X,f^{-1}(b_i)) = \chi(f^{-1}(b_i)) - \chi(f^{-1}(b_i^-)\cap B_{R_{b_i}}).$$

Therefore, we have

$$\chi(X) = \sum_{i=1}^{l}\Big(2\chi((f^{-1}(b_i)) - \chi(f^{-1}(b_i^-)\cap B_{R_{b_i}}) - \chi(f^{-1}(b_i^+)\cap B_{R_{b_i}})\Big)$$
$$+\sum_{i=1}^{l}\Big(\chi((f^{-1}(b_i^-)) + \chi((f^{-1}(b_i^+))\Big) - \sum_{i=1}^{l}\chi((f^{-1}(b_i)) - \sum_{i=1}^{l-1}\chi((f^{-1}(b_i^+))$$
$$= \sum_{i=1}^{l}\chi((f^{-1}(b_i)) + \sum_{i=1}^{m}\Big(\mathrm{ind}_\infty^+(f,X,f^{-1}(d_i)) + \mathrm{ind}_\infty^-(f,X,f^{-1}(d_i))\Big)$$
$$-\sum_{i=1}^{l-1}\chi((f^{-1}(b_i^+)) = \sum_{i=1}^{m}\Big(\mathrm{ind}_\infty^+(f,X,f^{-1}(d_i)) + \mathrm{ind}_\infty^-(f,X,f^{-1}(d_i))\Big)$$
$$+\int_{[b_1,b_l]}\chi(f^{-1}(t))d\chi_c(t).$$

To conclude, we remark that

$$\int_{[b_1,e_1[}\chi(f^{-1}(t))d\chi_c(t) = 0,$$

if $b_1 < e_1$ and

$$\int_{]e_l,b_l]}\chi(f^{-1}(t))d\chi_c(t) = 0,$$

if $e_l < b_l$.

□

5. Relations with functions with compact critical set

We recall that $\rho(x) = 1 + \frac{1}{2}(x_1^2 + \cdots + x_n^2)$ and that

$$\Gamma_{f,\rho} = \{x \in \mathbb{R}^n \mid \mathrm{rank}[\nabla f(x), \nabla\rho(x)] < 2\}.$$

Note that $\nabla\rho(x) = x$ and $\rho(x) \geq 1$. We have $\Sigma_f \subset \Gamma_{f,\rho}$.

Let $\Lambda_f = \{d_1, d_2 \ldots, d_m\}$.

Lemma 5.1. *There is $k \in \mathbb{N}$ such that for all $i \in \{1,2,\ldots,m\}$, for all $x \in \Gamma_{f,\rho} \setminus f^{-1}(d_i)$,*

$$|f(x) - d_i| > \frac{1}{\rho(x)^k},\ 1 \leq i \leq m,$$

for $|x| \gg 1$.

Proof. Note that 1 is the greatest critical value of ρ. We set $\tilde{S}_r = \rho^{-1}(r)$. Let $\beta :]1,+\infty[\to \mathbb{R}$ be defined by

$$\beta(r) = \inf\Big\{|f(x) - d_i| \mid x \in \tilde{S}_r \cap (\Gamma_{f,\rho} \setminus f^{-1}(d_i))\Big\}.$$

The function β is semi-algebraic. Furthermore $\beta > 0$ because for $r > 1$, $f_{|\tilde{S}_r}$ has a finite number of critical values. Thus the function $\frac{1}{\beta}$ is also semi-algebraic. Hence there exist $r_1 \geq 1$ and $k_0 \in \mathbb{N}$ such that $\frac{1}{\beta} < r^k$, for $r \geq r_1$ and $k \geq k_0$. This implies that $\beta(r) > \frac{1}{r^k}$ for $r \geq r_1$ and $k \geq k_0$. We can conclude that for $r \geq r_1$ and $k \geq k_0$,

$$|f(x) - d_i| > \frac{1}{\rho(x)^k},$$

for $x \in \tilde{S}_r \cap (\Gamma_{f,\rho} \setminus f^{-1}(d_i))$. □

Let $G_-(x) = F(x) - \frac{1}{\rho(x)^k}$ and let $g_- = G_{-|X}$.

Lemma 5.2. *We have* $\Lambda_f = \Lambda_{g_-}$

Proof. By definition of $g_-(x)$, we have that $\Gamma_{f,\rho} = \Gamma_{g_-,\rho}$. So if $\{x_n\}$ is a sequence of points in $\Gamma_{f,\rho}$ such that $\{x_n\} \to \infty$ then $\{f(x_n)\} \to d_i$ if and only if $\{g_-(x_n)\} \to d_i$ □

Lemma 5.3. *For* $R \gg 1$, $\chi(\{g_- \leq d_i\} \cap \tilde{S}_R) = \chi(\{f \leq d_i\} \cap \tilde{S}_R)$.

Proof. Let $R \gg 1$ be such that for all $x \in (\Gamma_{f,\rho} \setminus f^{-1}(d_i)) \cap \{\rho(x) \geq R\}$, $|f(x) - d_i| > \frac{1}{\rho(x)^k}$. Set $N_f^{\leq} = \{x \in \tilde{S}_R \mid f(x) \leq d_i\}$ and $N_{g_-}^{\leq} = \{x \in \tilde{S}_R \mid g_-(x) \leq d_i\}$. For $x \in \tilde{S}_R$, we have

$$g_-(x) \leq d_i \Leftrightarrow f(x) - \frac{1}{R^k} \leq d_i \Leftrightarrow f(x) \leq d_i + \frac{1}{R^k},$$

and so $N_f^{\leq} \subset N_{g_-}^{\leq}$. Furthermore if $0 < f(x) - d_i \leq \frac{1}{R^k}$ then $x \notin \Gamma_{f,\rho} \setminus f^{-1}(d_i)$ and therefore $\{f(x) \leq d_i + \frac{1}{R^k}\} \cap \tilde{S}_R$ retracts by deformation to $\{f(x) \leq d_i\} \cap \tilde{S}_R$. We get the result. □

Corollary 5.4. *We have* $\chi(\mathrm{Lk}^\infty(\{g_- \leq d_i\})) = \chi(\mathrm{Lk}^\infty(\{f \leq d_i\}))$.

Lemma 5.5. *Let* $\alpha \notin \Lambda_f$. *We have* $\chi(\mathrm{Lk}^\infty(\{g_- \leq \alpha\})) = \chi(\mathrm{Lk}^\infty(\{f \leq \alpha\}))$.

Proof. Let us study first the case when α belongs to an interval of $\mathbb{R} \setminus \Lambda_f$ bounded from above. We can assume that $0 \in \Lambda_f$ and that $b < 0$ is the greatest negative element of Λ_f (b can be $-\infty$).

Let α be such that $b < \alpha < 0$. We can find $R_b \gg 1$ such that $b < \frac{1}{2} + \frac{1}{R_b^k} < 0$. If $\{x_n\} \subseteq \Gamma_{g_-,\rho}$ is a sequence such that $b < g_-(x_n) \leq \frac{1}{2}\alpha$, then $\{g_-(x_n)\} \to b$. If $\rho(x_n) \geq R_b$ then $f(x_n) = g_-(x_n) + \frac{1}{\rho(x_n)^k} \leq g_-(x_n) + \frac{1}{R_b^k} \leq \frac{1}{2}\alpha + \frac{1}{R_b^k} < 0$. Then, $\{f(x_n)\}$ tend to b as well. As a consequence, there exists $R_0 \gg 1$ such that for all $R \geq R_0$ and $x \in \widetilde{S}_R \cap \Gamma_{g_-,\rho} \cap \{g_- \leq \frac{1}{2}\alpha\}$, $f(x) \leq \frac{b+\alpha}{2}$ and $g_-(x) \leq \frac{b+\alpha}{2}$.

To conclude, we have that $\mathrm{Lk}^\infty(\{g_- \leq \alpha\})$ is homeomorphic to $\{g_- \leq \alpha\} \cap \widetilde{S}_R$, $\mathrm{Lk}^\infty(\{f \leq \alpha\})$ is homeomorphic to $\{f \leq \alpha\} \cap \widetilde{S}_R$, and that

$$\{g_- \leq \alpha\} \cap \widetilde{S}_R = \{f \leq \alpha + \frac{1}{R^k}\} \cap \widetilde{S}_R$$

is homeomorphic to $\{f \leq \alpha\} \cap \widetilde{S}_R$, since $\widetilde{S}_R \cap \Gamma_{f,\rho} \cap \{\alpha \leq f \leq \alpha + \frac{1}{R^k}\} = \emptyset$.

Similarly if α belongs to the interval of $\mathbb{R} \setminus \Lambda_f$ not bounded from above, we can suppose that 0 is the biggest bifurcation value and that $\alpha > 0$. The proof is the same, replacing $\{g_- \leq \frac{b+\alpha}{2}\}$ with $\{g_- \geq \frac{\alpha}{2}\}$ and taking R such that $\alpha + \frac{1}{R^k} < 2\alpha$. □

Let $g_+(x) = f(x) + \frac{1}{\rho(x)^k}$. Note that $\Lambda_f = \Lambda_{g_+}$.

Lemma 5.6. *For* $R \gg 1$, $\chi(\{g_+ \geq d_i\} \cap \tilde{S}_R) = \chi(\{f \geq d_i\} \cap \tilde{S}_R)$.

Corollary 5.7. *We have* $\chi(\mathrm{Lk}^\infty(\{g_+ \geq d_i\})) = \chi(\mathrm{Lk}^\infty(\{f \geq d_i\}))$.

Lemma 5.8. *Let* $\alpha \notin \Lambda_f$. *We have* $\chi(\mathrm{Lk}^\infty(\{g_+ \geq \alpha\})) = \chi(\mathrm{Lk}^\infty(\{f \geq \alpha\}))$.

Lemma 5.9. *The sets* $(\nabla g_-)^{-1}(0)$ *and* $(\nabla g_+)^{-1}(0)$ *are compact.*

Proof. Let us suppose that $(\nabla g_-)^{-1}(0)$ is not compact. Therefore, there exists α a critical value of g_- such that $(\nabla g_-)^{-1}(0) \cap g_-^{-1}(\alpha)$ is not compact. Then, there exists $\{x_n\} \subset (\nabla g_-)^{-1}(0) \cap g_-^{-1}(\alpha)$ such that $\{x_n\} \to \infty$. Then $\{f(x_n)\} \to \alpha$ and $0 = \nabla f(x_n) + \frac{k}{\rho^{k+1}(x_n)} \nabla \rho(x_n)$. This implies that $x_n \in \Gamma_{f,\rho} \setminus f^{-1}(\alpha)$. Therefore α is a bifurcation value of f and $|f(x_n) - \alpha| = \frac{1}{\rho^k(x_n)}$, which contradicts Lemma 5.1. □

Corollary 5.10. *We have*

$$\int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{g_- \leq t\}))d\chi_c(t) = \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \leq t\}))d\chi_c(t),$$

and

$$\int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{g_+ \geq t\}))d\chi_c(t) = \int_{\mathbb{R}} \chi(\mathrm{Lk}^\infty(\{f \geq t\}))d\chi_c(t).$$

Corollary 5.11. *We have*

$$\int_{\mathbb{R}} \mathrm{ind}_g(f, X, f^{-1}(t))d\chi_c(t) = \int_{\mathbb{R}} \mathrm{ind}_g(g_-, X, g_-^{-1}(t))d\chi_c(t),$$

and

$$\int_{\mathbb{R}} \mathrm{ind}_g(-f, X, f^{-1}(t))d\chi_c(t) = \int_{\mathbb{R}} \mathrm{ind}_g(g_+, X, g_+^{-1}(t))d\chi_c(t).$$

If $X = \mathbb{R}^n$, we have that

$$\int_{\mathbb{R}} \mathrm{ind}_g(g_-, X, g_-^{-1}(t))d\chi_c(t) = \deg_\infty \nabla g_-,$$

so

$$\int_{\mathbb{R}} \mathrm{ind}_g(f, X, f^{-1}(t))d\chi_c(t) = \deg_\infty \nabla g_-.$$

Moreover, if W_- is the vector field defined by $W_- = \rho^{k+1}\nabla f + \nabla \rho$, then $\deg_\infty W_- = \deg_\infty \nabla g_-$ and so,

$$\int_{\mathbb{R}} \mathrm{ind}_g(f, X, f^{-1}(t))d\chi_c(t) = \deg_\infty W_-.$$

We can apply the same procedure to g_+ and obtain a vector field W_+. We note if f is a polynomial then W_- and W_+ are polynomial vector fields. This gives global versions of results of [6] and [11].

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